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ON SOME SUBCLASSES OF MULTIVALENT STARLIKE FUNCTIONS

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الملخص

في هذه الورقة البحثية نقدم فصلين جزئيين جديدين $T^{*}_{m.\lambda.\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ و $C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ من الدوال النجمية متعددة التكافؤ في قرص الوحدة المفتوح معرفين باستخدام مؤثرمعين من حساب التفاضل و التكامل الكسري. سوف نحصل على متباينات المعامل وخواص التشوه للدوال المنتمية إلى الفصلين الجزئيين أعلاه.بالإضافة إلى تحديد أنصاف أقطار التحدب والنجمية للدوال المنتمية إلى هاتين الفصلين الجزئيين.

Abstract

In the present paper we are introducing two new subclasses $T^{*}_{m.\lambda.\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ and $C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ of multivalent starlike functions in the open unit disk defined by using certain fractional calculus operator. We obtain coefficient inequalities and distortion properties for functions belonging to the above subclasses. The radii of convexity and starlikeness for functions belonging to these subclasses are also determined.

Keywords: multivalent (or p-valent) functions, starlike functions, convex functions, fractional derivative operator.

1- Introduction and Definitions

Let $A(p)$ denote the class of functions defined by

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}\,z^{p+n}(p\in\mathbb{N}). \tag{1.1}$$

which are analytic and multivalent (or p-valent) in the open unit disk $\mathcal{U}=\{z:|z|<1\}$. A function $f(z)\in A(p)$ is called p-valent starlike of order α if $f(z)$ satisfies the conditions

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha (1.2)$$

and

$$\int_{0}^{2\pi}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}d\theta=2p\pi\,(1.3)$$

for $0 \leq \alpha < p,\ p \in \mathbb{N}$, and $z \in \mathcal{U}$. We denote by $S^*(p,\alpha)$ the class of all p-valent starlike functions of order α. Also a function $f(z) \in A(p)$ is called p-valent convex of order α if $f(z)$ satisfies the following conditions

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (1.4)$$

and

$$\int_0^{2\pi} \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi \quad (1.5)$$

for $0 \leq \alpha < p,\ p \in \mathbb{N}$, and $z \in \mathcal{U}$. We denote by $K(p,\alpha)$ the class of all p-valent convex functions of order α. We note that

$$f(z) \in K(p,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S^*(p,\alpha) \quad (1.6)$$

for $0 \leq \alpha < p$.

The class $S^*(p,\alpha)$ was introduced by Patil and Thakare [7], and the class $K(p,\alpha)$ was introduced by Owa [6]. In particular, the classes $S^*(1,\alpha) = S^*(\alpha)$ and $K(1,\alpha) = K(\alpha)$ when $p = 1$ were studied by Silverman [8].

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \left(a_{p+n} \geq 0, p \in \mathbb{N}\right). \quad (1.7)$$

We denote by $T^*(p,\alpha)$ and $C(p,\alpha)$, the classes obtained by taking intersections, respectively, of the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ with the class $T(p)$, that is,

$$T^*(p,\alpha) = S^*(p,\alpha) \cap T(p),$$

and

$$C(p,\alpha) = K(p,\alpha) \cap T(p).$$

The classes $T^*(p,\alpha)$ and $C(p,\alpha)$ were introduced by Owa [6]. In particular, the classes $T^*(1,\alpha) = T^*(\alpha)$ and $C(1,\alpha) = C(\alpha)$ when $p = 1$ were studied by Silverman [8].

Let ${}_2F_1(a,b;\ c;\ z)$be the Gauss hypergeometric function defined for $z \in \mathcal{U}$by, (see Srivastava and Karlsson [9])

$${}_2F_1(a,b;\ c;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!} z^n$$

where $(\lambda)_n$ is the Pochhammer symbole defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+n-1), & n \in \mathbb{N}. \end{cases}$$

for $\lambda \neq 0, -1, -2, \dots$.

We recall the following definitions of fractional derivative operators which were used by Owa [5], (see also [10]) as follows:

Definition 1.1. The fractional derivative operator of order λ is defined by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^\lambda} d\xi, \qquad 0 \leq \lambda < 1 \tag{1.8}$$

where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 1.2. Let $0 \leq \lambda < 1$, and$\mu, \eta \in \mathbb{R}$.Then, in terms of the familiar Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1 - \frac{\xi}{Z}\right) d\xi\right) \tag{1.9}$$

where $f(z)$ is analytic function in a simply- connected region of the z-plane containing the origin, with the order $f(z) = O(|z|^\varepsilon), z \to 0$, where $\varepsilon >$max$\{0, \mu - \eta\} - 1$ and the multiplicity of $(z-\xi)^{-\lambda}$ is removed requiring $\log(z-\xi)$ to be real when $z - \xi > 0$.
Notice that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^\lambda f(z), \qquad 0 \leq \lambda < 1 \tag{1.10}$$

The fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta,p} f(z)$ which maps $A(p)$ into itself was studied by Amsheri and Zharkova [1], (see also [3], [13])defined by

$$M_{0,z}^{\lambda,\mu,\eta,p} f(z) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^\mu J_{0,z}^{\lambda,\mu,\eta} f(z)$$

$$= z^p + \sum_{n=1}^{\infty} \gamma_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n} \qquad (1.11)$$

for $f(z) \in A(p)$ and $\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda, \mu) - p - 1; p \in \mathbb{N}$, where

$$\gamma_n(\lambda, \mu, \eta, p) = \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n}\ , \qquad (n \in \mathbb{N}) \qquad (1.12)$$

Corresponding to the operator $M_{0,z}^{\lambda,\mu,\eta,p}$ defined in (1.11), Zayed et al. [13] introduced the operator$N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}(z) : A(p) \to A(p)$, for $m \in \mathbb{N}_o = \mathbb{N} \cup \{0\}$ and $\delta \geq 0$ defined by:

$$N_{0,z}^{0,\lambda,\mu,\eta,\delta,p} f(z) = M_{0,z}^{\lambda,\mu,\eta,p} f(z)$$

$$N_{0,z}^{1,\lambda,\mu,\eta,\delta,p} f(z) = N_{0,z}^{\lambda,\mu,\eta,\delta,p} f(z)$$

$$= (1-\delta) M_{0,z}^{\lambda,\mu,\eta,p} f(z) + \delta \frac{z}{p} \left(M_{0,z}^{\lambda,\mu,\eta,p} f(z) \right)'$$

$$= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p} \right) \gamma_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n}$$

and (in general),

$$N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) = N_{0,z}^{\lambda,\mu,\eta,\delta,p} \left(N_{0,z}^{m-1,\lambda,\mu,\eta,\delta,p} f(z) \right)$$

$$= z^p + \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p} \right)^m \gamma_n(\lambda, \mu, \eta, p) a_{p+n} z^{p+n} \qquad (1.13)$$

Motivated essentially by aforementioned works, we introduce two new subclasses $T^*_{m,\lambda,\mu,\eta,\delta}(p, \alpha, \beta, \sigma)$ and $C_{m,\lambda,\mu,\eta,\delta}(p, \alpha, \beta, \sigma)$of analytic and multivalent starlike functions $f(z)$ belonging to the class $T(p)$ involving certain fractional derivative operator as follows:

Definition 1.3. The function $f(z) \in T(p)$ is said to be in the class $T^*_{m,\lambda,\mu,\eta,\delta}(p, \alpha, \beta, \sigma)$ if

$$\left| \frac{\dfrac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)'}{g(z)} - p}{\dfrac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)\right)'}{g(z)} + p - 2\beta} \right| < \sigma \qquad (z \in \mathcal{U}) \qquad (1.14)$$

$(m \in \mathbb{N}_0;\ \lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda, \mu) - p - 1; 0 \leq \alpha < p; 0 \leq \beta < p; 0 < \sigma \leq 1; \delta \geq 0; p \in \mathbb{N})$

For the function

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \left(b_{p+n} \geq 0, p \in \mathbb{N}\right) \tag{1.15}$$

belonging to the class $T^*(p,\alpha)$. Denoted by $N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z)$ the extension of the fractional derivative operator which is defined for $f(z) \in T(p)$ by$N_{0,z}^{m,\lambda,\mu,\eta,\delta,p} f(z) =$

$$z^p - \sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n} \left(a_{p+n} \geq 0\right) \tag{1.16}$$

Further, if $f(z) \in T(p)$ satisfies the condition (1.14) for $g(z) \in C(p,\alpha)$, we say that $f(z) \in C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$.

The above-defined classes $T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ and $C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$are of special interest and they contain many well-known classes of analytic functions. In particular, for $m = 0$, we have

$$T^*_{0,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma) = T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\sigma)$$

and

$$C_{0,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma) = C_{\lambda,\mu,\eta}(p,\alpha,\beta,\sigma)$$

where $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\sigma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\sigma)$ are precisely the subclasses of p-valent starlike functions which were studied by Amsheri and Zharkova[1].

Furthermore, for $\lambda = \mu = \delta = 0$, we have

$$T^*_{m,0,0,\eta,0}(p,\alpha,\beta,\sigma) = T^*(p,\alpha,\beta,\sigma)$$

and

$$C_{m,0,0,\eta,0}(p,\alpha,\beta,\sigma) = C(p,\alpha,\beta,\sigma)$$

where $T^*(p,\alpha,\beta,\sigma)$ and $C(p,\alpha,\beta,\sigma)$ are precisely the subclasses of p-valent starlike functions which were studied by Aouf and Hossen [2]. Thus, for $\lambda = \mu = \delta = 0$ and $p = 1$, we have

$$T^*_{m,0,0,\eta,0}(1,\alpha,\beta,\sigma) = T^*(\alpha,\beta,\sigma)$$

and

$$C_{m,0,0,\eta,0}(1,\alpha,\beta,\sigma) = C(\alpha,\beta,\sigma)$$

where $T^*(\alpha,\beta,\sigma)$ and $C(\alpha,\beta,\sigma)$ are the subclasses of starlike functions which were studied by Srivastava and Owa [11], [12].

In this paper, sharp results concerning coefficients, distortion theorems and radii of convexity and starlikeness for functions belonging to the classes $T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ and $C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ are determined.

In order to prove our results we shall need the following lemmas for the classes $T^*(p,\alpha)$ and $Cp,\alpha)$ due to Owa [6].

Lemma 1.4. Let the function $g(z)$ defined by (1.15). Then $g(z)$ is in the class $T^*(p,\alpha)$ if and only if

$$\sum_{n=1}^{\infty}(p+n-\alpha)b_{p+n} \le (p-\alpha) \qquad (1.17)$$

Lemma 1.5. Let the function $g(z)$ defined by (1.15). Then $g(z)$ is in the class $C(p,\alpha)$ if and only if

$$\sum_{n=1}^{\infty}(p+n)(p+n-\alpha)b_{p+n} \le p(p-\alpha) \qquad (1.18)$$

2- Coefficient Inequalities

Theorem 2.1. Let the function $f(z)$ be defined by (1.7). If $f(z)$ belongs to the class $T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$, then

$$\sum_{n=1}^{\infty}\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)(1+\sigma)a_{p+n} - \frac{(p-\alpha)[p(1-\sigma)+2\sigma\beta]}{(p+n-\alpha)} \le 2\sigma(p-\beta) \quad (2.1)$$

where$\gamma_n(\lambda,\mu,\eta,p)$ is given by (1.12).

Proof. Since $(z) \in T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$, there exist a function $g(z)$ belonging to the class $T^*(p,\alpha)$ such that

$$\left|\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)' - pg(z)}{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)' + (p-2\beta)g(z)}\right| < \sigma \qquad z \in \mathcal{U} \qquad (2.2)$$

It follows from (2.2) that

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}\left[\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)a_{p+n} - pb_{p+n}\right]z^n}{2(p-\beta) - \sum_{n=1}^{\infty}\left[\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)a_{p+n} + (p-2\beta)b_{p+n}\right]z^n}\right\} < \sigma \quad (2.3)$$

Choosing values of z on the real axis so that $\frac{z\left(N_{0,z}^{m,\lambda,\mu,\eta,\delta,p}f(z)\right)'}{g(z)}$ is real, and letting $z \to 1^-$ through real axis, we have

$$\sum_{n=1}^{\infty}\left[\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)a_{p+n} - pb_{p+n}\right]$$

$$\leq \sigma\left\{2(p-\beta) - \sum_{n=1}^{\infty}\left[\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)a_{p+n} + (p-2\beta)b_{p+n}\right]\right\}$$

or, equivalently,

$$\sum_{n=1}^{\infty}\left\{\left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)(1+\sigma)a_{p+n} - [p(1-\sigma)+2\sigma\beta]b_{p+n}\right\}$$

$$\leq 2\sigma(p-\beta) \qquad (2.4)$$

Note that, by using Lemma 1.4, $g(z) \in T^*(p,\alpha)$ implies

$$b_{p+n} \leq \frac{p-\alpha}{p+n-\alpha} \qquad (2.5)$$

Making use of (2.5) in (2.4), we complete the proof of Theorem 2.1.

Corollary 2.2. Let the function $f(z)$ be defined by (1.7) in the class $T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$. Then

$$a_{p+n} \leq \frac{2\sigma(p-\beta)(p+n-\alpha) + (p-\alpha)[p(1-\sigma)+2\sigma\beta]}{\gamma_n(\lambda,\mu,\eta,p)(p+n)(1+\sigma)(p+n-\alpha)\left(\frac{p+\delta n}{p}\right)^m} \qquad (2.6)$$

where $\gamma_n(\lambda,\mu,\eta,p)$is given by (1.12).The result (2.6) is sharp for a function of the form:

$$f(z) = z^p - \frac{2\sigma(p-\beta)(p+n-\alpha) + (p-\alpha)[p(1-\sigma)+2\sigma\beta]}{\gamma_n(\lambda,\mu,\eta,p)(p+n)(1+\sigma)(p+n-\alpha)\left(\frac{p+\delta n}{p}\right)^m} z^{p+n} \qquad (2.7)$$

with respect to

$$g(z) = z^p - \frac{p-\alpha}{p+n-\alpha} z^{p+n} (\, n \geq 1) \qquad (2.8)$$

Remark 1.

By specializing the parameters $p, \lambda, \mu, \alpha, \delta$ and m we obtain the following results which were studied by various other authors:

1- Letting $p = 1, \lambda = \mu = \delta = 0$, and $\alpha = 0$ in corollary 2.2, we obtain a result was proved by [Gupta [4], Theorem 3].
2- Letting $\lambda = \mu = \delta = 0$in Theorem 2.1 and Corollary 2.2 respectively, we obtain results were proved by [Aouf and Hossen [2], in Theorem 1 and Corollary 2].
3- Letting $m = 0$ in Theorem 2.1 and Corollary 2.2 respectively, we obtain results were proved by [Amsheri and Zharkova [1], in Theorem 3.1 and Corollary 3.2].

In a similar manner, Lemma1.5 can be used to prove the following Theorem:

Theorem 2.3. Let the function $f(z)$ be defined by (1.7). If $f(z)$ belongs to the class $C_{\lambda,\mu,\eta,\delta}(p, \alpha, \beta, \sigma)$. Then

$$\sum_{n=1}^{\infty} \left(\frac{p+\delta n}{p}\right)^m \gamma_n(\lambda,\mu,\eta,p)(p+n)(1+\sigma)a_{p+n} - \frac{p(p-\alpha)[p(1-\sigma)+2\sigma\beta]}{(p+n)(p+n-\alpha)} \leq 2\sigma(p-\beta) \qquad (2.9)$$

where$\gamma_n(\lambda,\mu,\eta,p)$is given by(1.12).

Corollary 2.4. Let the function $f(z)$ be defined by (1.7) in the class $C_{\lambda,\mu,\eta,\delta}(p, \alpha, \beta, \sigma)$. Then

$$a_{p+n} \leq \frac{2\sigma(p-\beta)(p+n)(p+n-\alpha) + p(p-\alpha)[p(1-\sigma)+2\sigma\beta]}{\gamma_n(\lambda,\mu,\eta,p)(p+n)^2(1+\sigma)(p+n-\alpha)\left(\frac{p+\delta n}{p}\right)^m} \qquad (2.10)$$

where $\gamma_n(\lambda,\mu,\eta,p)$is given by (1.12). The result (2.10) is sharp for a function of the form:

$$f(z) = z^p - \frac{2\sigma(p-\beta)(p+n)(p+n-\alpha) + p(p-\alpha)[p(1-\sigma)+2\sigma\beta]}{\gamma_n(\lambda,\mu,\eta,p)(p+n)^2(1+\sigma)(p+n-\alpha)\left(\frac{p+\delta n}{p}\right)^m} z^{p+n} \qquad (2.11)$$

with respect to

$$g(z) = z^p - \frac{p(p-\alpha)}{(p+n)(p+n-\alpha)} z^{p+n} \qquad (n \geq 1) \qquad (2.12)$$

3- Distortion properties

Next, we state and prove results concerning distortion properties of $f(z)$ belonging to the classes $T^*_{m,\lambda,\mu,\eta,\delta}(p, \alpha, \beta, \sigma)$ and $C_{m,\lambda,\mu,\eta,\delta}(p, \alpha, \beta, \sigma)$.

Theorem 3.1. let $\lambda, \mu, \eta \in \mathbb{R}$,such that

$$\lambda \geq 0, \mu < p+1, \eta \geq \lambda\left(1-\frac{p+2}{\mu}\right), \delta \geq 0, m \in \mathbb{N}_0 \text{ and } p \in \mathbb{N} \quad (3.1)$$

Also, let $f(z)$ defined by (1.7) be in the class $T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$. Then

$$|f(z)| \geq |z|^p - A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)|z|^{p+1}, \quad (3.2)$$

$$|f(z)| \leq |z|^p + A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)|z|^{p+1}, \quad (3.3)$$

$$|f'(z)| \geq p|z|^{p-1} - (p+1)A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)|z|^{p}, \quad (3.4)$$

and

$$|f'(z)| \leq p|z|^{p-1} + (p+1)A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)|z|^{p}, \quad (3.5)$$

forz $\in \mathcal{U}$, provided that $0 \leq \alpha < p, 0 \leq \beta < p,$ and $0 < \sigma \leq 1$, where

$$A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma) = \frac{(1+p-\mu)(1+p+\eta-\lambda)[2\sigma(p-\beta)+p(p-\alpha)(1+\sigma)]}{(1+p+\eta-\mu)(p+1)^2(1+\sigma)(p+1-\alpha)\left(\frac{p+\delta}{p}\right)^m} \quad (3.6)$$

The estimates for$|f(z)|$ and $|f'(z)|$ are sharp.

Proof. We observe that the function $\gamma_n(\lambda,\mu,\eta,p)$ defined by (1.12) satisfy the inequality$\gamma_n(\lambda,\mu,\eta,p) \leq \gamma_{n+1}(\lambda,\mu,\eta,p)$, $\forall n \in \mathbb{N}$, provided that $\eta \geq \lambda\left(1-\frac{p+2}{\mu}\right)$. Thereby, showing that $\gamma_n(\lambda,\mu,\eta,p)$ is non-decreasing. Thus under the conditions stated in (3.1), we have

$$0 < \frac{(1+p)(1+p+\eta-\mu)}{(1+p-\mu)(1+p+\eta-\lambda)} = \gamma_1(\lambda,\mu,\eta,p) \leq \gamma_n(\lambda,\mu,\eta,p) \forall n \in \mathbb{N} \quad (3.7)$$

For $f(z) \in T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$, (2.4) implies

$$\left(\frac{p+\delta}{p}\right)^m \gamma_1(\lambda,\mu,\eta,p)(p+1)(1+\sigma)\sum_{n=1}^{\infty} a_{p+n} - [p(1-\sigma)+2\sigma\beta]\sum_{n=1}^{\infty} b_{p+n} \leq 2\sigma(p-\beta) (3.8)$$

For $g(z) \in T^*(p,\alpha)$, Lamma 1.4 yields

$$\sum_{n=1}^{\infty} b_{p+n} \leq \frac{p-\alpha}{p+1-\alpha} \quad (3.9)$$

So that (3.8) reduces to

$$\sum_{n=1}^{\infty} a_{p+n} \le \frac{(1+p-\mu)(1+p+\eta-\lambda)[2\sigma(p-\beta)+p(p-\alpha)(1+\sigma)]}{(1+p+\eta-\mu)(p+1)^2(1+\sigma)(p+1-\alpha)\left(\frac{p+\delta}{p}\right)^m}$$

$$= A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma) \qquad (3.10)$$

Consequently,

$$|f(z)| \ge |z|^p - |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n} \quad (3.11)$$

and

$$|f(z)| \le |z|^p + |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n} \quad (3.12)$$

By using (3.11), (3.12) and (3.10), we easily arrive at the desired results (3.2) and (3.3).

Furthermore, we note from (2.4) that

$$\left(\frac{p+\delta}{p}\right)^m \gamma_1(\lambda,\mu,\eta,p)(1+\sigma)\sum_{n=1}^{\infty}(p+n)a_{p+n} - [p(1-\sigma)+2\sigma\beta]\sum_{n=1}^{\infty} b_{p+n}$$

$$\le 2\sigma(p-\beta) (3.13)$$

which in view of (3.9), becomes

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \le \frac{(1+p-\mu)(1+p+\eta-\lambda)[2\sigma(p-\beta)+p(p-\alpha)(1+\sigma)]}{(1+p+\eta-\mu)(p+1)(1+\sigma)(p+1-\alpha)\left(\frac{p+\delta}{p}\right)^m}$$

$$= (p+1)A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma) \qquad (3.14)$$

Thus, we have

$$|f'(z)| \ge p|z|^{p-1} - |z|^p\sum_{n=1}^{\infty}(p+n)a_{p+n} \quad (3.15)$$

and

$$|f'(z)| \le p|z|^{p-1} + |z|^p\sum_{n=1}^{\infty}(p+n)a_{p+n} \quad (3.16)$$

On using (3.15),(3.16) and (3.14), we arrive at the desired results (3.4) and (3.5).

Finally, we can prove that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp by taking the function

$$f(z) = z^p - \frac{(1+p-\mu)(1+p+\eta-\lambda)[2\sigma(p-\beta)+p(p-\alpha)(1+\sigma)]}{(1+p+\eta-\mu)(p+1)^2(1+\sigma)(p+1-\alpha)\left(\frac{p+\delta}{p}\right)^m} z^{p+1} \tag{3.17}$$

with respect to

$$g(z) = z^p - \frac{(p-\alpha)}{(p+1-\alpha)} z^{p+1} \tag{3.18}$$

This completes the proof of Theorem 3.1.

Corollary 3.2. Let the function $f(z)$ be defined by (1.7) in the class $T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| \le r_1$, where

$$r_1 = 1 - \frac{(1+p-\mu)(1+p+\eta-\lambda)[2\sigma(p-\beta)+p(p-\alpha)(1+\sigma)]}{(1+p+\eta-\mu)(p+1)^2(1+\sigma)(p+1-\alpha)\left(\frac{p+\delta}{p}\right)^m} \tag{3.19}$$

The result is sharp with the extremal function defined by (3.17).

Remark 2.

By specializing the parameters $p, \lambda, \mu, \alpha, \delta$ and m we obtain the following results which were studied by various other authors:

1- Letting $p = 1$, $\lambda = \mu = \delta = 0$, and $\alpha = 0$ in Theorem 3.1, we obtain a result was proved by [Gupta [4], Theorem4].
2- Letting $\lambda = \mu = \delta = 0$ in Theorem 3.1 and Corollary 3.2 respectively, we obtain results were proved by [Aouf and Hossen [2], in Theorem 3 and Corollary 3].
3- Letting $m = 0$ in Theorem 3.1 and Corollary 3.2 respectively, we obtain results were proved by [Amsheri and Zharkova [1], in Theorem 4.1 and Corollary 4.2].

Theorem 3.3. Under the conditions stated in (3.1), let the function $f(z)$ defined by (1.7) be in the class $C_{\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$, then

$$|f(z)| \ge |z|^p - B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)|z|^{p+1}, \tag{3.20}$$

$$|f(z)| \le |z|^p + B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)|z|^{p+1}, \tag{3.21}$$

$$|f'(z)| \ge p|z|^{p-1} - (p+1)B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)|z|^p, \tag{3.22}$$

and

$$|f'(z)| \leq p|z|^{p-1} + (p+1)B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)|z|^p, \tag{3.23}$$

forz $\in \mathcal{U}$, provided that $0 \leq \alpha < p, 0 \leq \beta < p,$ and $0 < \sigma \leq 1$, where

$$B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma) = (1+p-\mu)(1+p+\eta-\lambda)$$

$$\times \left(\frac{2\sigma(p-\beta)(p+1)(p+1-\alpha) + p(p-\alpha)[p(1-\sigma)+2\sigma\beta]}{(1+p+\eta-\mu)(p+1)^3(1+\sigma)(p+1-\alpha)\left(\frac{p+\delta}{p}\right)^m} \right) \tag{3.24}$$

The estimates for$|f(z)|$ and $|f'(z)|$ are sharp.

Proof. By using Lemma 1.5, we have

$$\sum_{n=1}^{\infty} b_{p+n} \leq \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} \tag{3.25}$$

Since $g(z) \in C(p,\alpha)$, the assertions (3.20), (3.21), (3.22) and (3.23) of theorem 3.3 follow if we apply (3.25) to (2.4)
The estimates for$|f(z)|$ and $|f'(z)|$ are attained by the function

$$f(z) = z^p - (1+p-\mu)(1+p+\eta-\lambda)$$

$$\times \left(\frac{2\sigma(p-\beta)(p+1)(p+1-\alpha) + p(p-\alpha)[p(1-\sigma)+2\sigma\beta]}{(1+p+\eta-\mu)(p+1)^3(1+\sigma)(p+1-\alpha)\left(\frac{p+\delta}{p}\right)^m} \right) z^{p+1} \tag{3.26}$$

with respect to

$$g(z) = z^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} z^{p+1} \tag{3.27}$$

This completes the proof of Theorem 3.3.

Corollary 3.4. Let the function $f(z)$ be defined by (1.7) be in the class$C_{\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| < r_2$, where

$$r_2 = 1 - (1+p-\mu)(1+p+\eta-\lambda)$$

$$\times\left(\frac{2\sigma(p-\beta)(p+1)(p+1-\alpha)+p(p-\alpha)[p(1-\sigma)+2\sigma\beta]}{(1+p+\eta-\mu)(p+1)^3(1+\sigma)(p+1-\alpha)\left(\frac{p+\delta}{p}\right)^m}\right) \quad (3.28)$$

The result is sharp with the extremal function defined by (3.26).

4- Radii of convexity and starlikeness

In view of lemma1.4, we know that the function $f(z)$ defined by (1.7) is p-valent starlike in the unit disk $\mathcal{U}$ if and only if

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \leq p \quad (4.1)$$

for $f(z) \in T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$, we find from (2.4) and (3.9)

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \leq (p+1)A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma) \leq p \quad (4.2)$$

where$A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ is defined by (3.6).

Furthermore, for $f(z) \in C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$, we have

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \leq (p+1)\, B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma) \leq p \quad (4.3)$$

where $B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ is defined by (3.24). Thus we observe that $T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ and $C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ are subclasses of p- valent starlike functions. Naturally, therefore, we are interested in finding the radii of convexity and starlikeness for functions in$T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ and $C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$.We first state:

Theorem 4.1. Let the function $f(z)$ defined by (1.7) be in the class $T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$.Then $f(z)$ is p-valent convex in the disk $|z| < r_3$, where

$$r_3 = \inf_{n\in\mathbb{N}}\left\{\frac{p^2}{(p+1)(p+n)A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)}\right\}^{1/n} \quad (4.4)$$

and $A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ is given by (3.6). The result is sharp

proof. It suffices to prove

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \leq p \qquad (|z| < r_3) \quad (4.5)$$

Indeed we have

$$\left|1+\frac{z f''(z)}{f'(z)}-p\right| = \left|\frac{-\sum_{n=1}^{\infty} n(p+n)a_{p+n}z^n}{p-\sum_{n=1}^{\infty}(p+n)a_{p+n}z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}|z|^n}{p-\sum_{n=1}^{\infty}(p+n)a_{p+n}|z|^n} \tag{4.6}$$

Hence (4.5) is true if

$$\sum_{n=1}^{\infty} n(p+n)a_{p+n}|z|^n \leq p^2 - \sum_{n=1}^{\infty} p(p+n)a_{p+n}|z|^n \tag{4.7}$$

That is, if

$$\sum_{n=1}^{\infty}(p+n)^2 a_{p+n}|z|^n \leq p^2 \tag{4.8}$$

With the aid of (3.14),(4.8) is true if

$$(p+n)|z|^n \leq \frac{p^2}{(p+1)A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)} \tag{4.9}$$

Solving (4.9) for $|z|$, we get

$$|z| \leq \left\{\frac{p^2}{(p+1)(p+n)A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)}\right\}^{1/n} \quad (n \geq 1) \tag{4.10}$$

Finally, since $(p+n)^{-1/n}$ is an increasing function for integers $n \geq 1,\ p \in \mathbb{N}$, we have (4.5) for $|z| < r_3$, where r_3 is given by (4.4).

In order to complete the proof of Theorem 4.1, we note that the result is sharp for the function $f(z) \in T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ of the form:

$$f(z) = z^p - \frac{(p+1)A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)}{(p+n)} z^{p+n} \tag{4.11}$$

for some integers $n \geq 1$.

Similarly we can prove the next theorem.

Theorem 4.2. Let the function $f(z)$ defined by (1.7) be in the class $C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$.Then $f(z)$ is p-valent convex in the disk $|z| < r_4$, where

$$r_4 = \inf_{n\in\mathbb{N}} \left\{\frac{p^2}{(p+1)(p+n)\, B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)}\right\}^{1/n} \tag{4.12}$$

and $B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ is given by (3.24). The result is sharp for the function $f(z) \in C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ of the form:

$$f(z) = z^p - \frac{(p+1)\, B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)}{(p+n)} z^{p+n} \tag{4.13}$$

for some integers $n \geq 1$.

Remark 3.

By specializing the parameters λ, μ, δ and m we obtain the following results which were studied by other authors:

1- Letting $\lambda = \mu = \delta = 0$in Theorem 4.1 and Theorem 4.2 respectively, we obtain results were proved by [Aouf and Hossen [2], in Theorem 5 and Theorem 6].

2- Letting $m = 0$ in Theorem 4.1 and Theorem 4.2 respectively, we obtain results were proved by [Amsheri and Zharkova [1], in Theorem 6.1 and Theorem 6.2].

Theorem 4.3. Let the function $f(z)$ defined by (1.7) be in the class $T^*_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$.Then $f(z)$ is p-valent starlike in the disk $|z| < r_5$, where

$$r_5 = \inf_{n\in\mathbb{N}} \left\{ \frac{p}{(p+1)A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)} \right\}^{1/n} \tag{4.14}$$

and $A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ is given by (3.6). The result is sharp with the extermal function $f(z)$ given by (4.11).

proof. It suffices to prove

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p \qquad (|z| < r_5) \tag{4.15}$$

Indeed we have

$$\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{-\sum_{n=1}^{\infty} na_{p+n} z^n}{1 - \sum_{n=1}^{\infty} a_{p+n} z^n} \right|$$

$$\leq \frac{\sum_{n=1}^{\infty} na_{p+n} |z|^n}{1 - \sum_{n=1}^{\infty} a_{p+n} |z|^n} \tag{4.16}$$

Hence (4.15) is true if

$$\sum_{n=1}^{\infty} na_{p+n} |z|^n \leq p - \sum_{n=1}^{\infty} pa_{p+n} |z|^n \tag{4.17}$$

That is, if

$$\sum_{n=1}^{\infty} \frac{(p+n)}{p} a_{p+n} |z|^n \leq 1 \tag{4.18}$$

With the aid of (3.14),(4.18) is true if

$$\frac{1}{p} |z|^n \leq \frac{1}{(p+1) A_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)} \tag{4.19}$$

Solving (4.19) for $|z|$, we getthe desired result (4.14).

Similarly we can prove the next theorem.

Theorem 4.4. Let the function $f(z)$ defined by (1.7) be in the class $C_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$.Then $f(z)$ is p-valent starlike in the disk $|z| < r_6$, where

$$r_6 = \inf_{n \in \mathbb{N}} \left\{ \frac{p}{(p+1)\, B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)} \right\}^{1/n} \tag{4.20}$$

and $B_{m,\lambda,\mu,\eta,\delta}(p,\alpha,\beta,\sigma)$ is given by (3.24). The result is sharp with the extermal function $f(z)$ given by (4.13).

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