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NUMERICAL SOLUTIONS OF BERNOULLI DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVESBY RUNGE-KUTTA TECHNIQUES

Mufeedah Maamar Salih Ahmed
Department of Mathematics, Faculty of Art & Science Kasr Khiar
Elmergib University, Khums, Libya
mmsahmad32@gmail.com

Abstract
In this article, we are discussing the numerical solution of Brnoulli's equation with fractional derivatives subject to initial value problems by applying 4th order Runge-Kutta, modified Runge-Kutta and Runge-Kutta Mersian methods. Here the solutions of some numerical examples have been obtained with the help of mathematica program as well as we determined the exact analytic solutions.
Keywords: Bernoulli equation with fractional derivatives, Initial value problem, Runge-Kutta, Modified Runge-Kutta and Runge-Kutta Mersian Methods.

الملخص

في هذه المقالة، ناقشنا الحل العددي لمعادلة برنولي مع المشتقات الكسرية الخاضعة لمسائل القيمة الأولية من خلال تطبيق طرق Runge-Kutta من الدرجة الرابعة و Runge-Kutta و Runge-Kutta Mersianالمعدلة. هنا تم الحصول على حلول لبعض الأمثلة العددية بمساعدة برنامج mathematica وكذلك قمنا بتحديد الحلول التحليلية الدقيقة.

الكلمات المفتاحية: معادلة برنولي مع المشتقات الكسرية، مشكلة القيمة الأولية ، طرق رونج-كوتا ، طرق رونج-كوتا المعدلة وطرق رونج-كوتا ميرسيان.

1. Introduction
The differential equations are the most important mathematical model of physical phenomenon. Many applications of differential equations, particularly ordinary differential equations of different orders, can be found in the mathematical modeling of real life problems. Most of models of these problems formulated by means of these equations are so complicated to determine the exact solution and one of two approaches is taken to approximate solution. Therefore, many theoretical and numerical studies dealing with the solution of such differential equations of different order have appeared in the literature. Thus, there are many analytical and numerical methods for solving some types of the differential equations. Now, the fractional differential equations is a generalization of ordinary differential equations, and differential equations with fractional order derivative have recently proven to be strong tools in the modeling of many physical phenomena and in various fields of science and engineering.

(see [1],[5],[7]) There has been a significant development in ordinary and partial fractional differential equations with fractional order in recent years.

Many researchers developed the family of Runge-Kutta methods for solving first, second and third order ordinary differential equations, For example [18] has developed a singly diagonally implicit Runge-Kutta-Nyström method for second-order ordinary differential equations with periodical solutions. Many applications have been solved base Runge Kutta methods. [7] Solved discrete-time model representation for biochemical pathway systems based on Runge–Kutta method. In [19], derived some efficient methods for solving second order ordinary differential equations, which have oscillating solutions, furthermore, it is essential to consider the phase-lag and the dissipation error that result from comparing.

Theordinary differential equation can be solve by using multistep methods, this methods it would be more efficient in case higher order ODEs can be solved using special numerical methods, (see [4,11-13]).

In ([2], [3]), Alonso-Mallo and Cano have developed and analyzed a technique which can be used in Runge-Kutta or Rosenbrock methods to avoid such order reduction. Such methods provide strong reductions of computational cost with respect to other classical, explicit or implicit methods.The authors in [10] studied unconditional stability properties of explicit exponential Runge Kutta methods when they are applied to semi-linear systems of ODEs characterized by a stiff linear systems f stiff nonlinear part.

2. Preliminary Material on Fractional Calculus

In this section, some we review of the helpful definitions in fractional calculus, and we recall the properties that we will use in the subsequent sections. For a more comprehensive introduction to this subject, the reader can be the see referred: [6, 14-17].

We consider the Riemann–Liouville (RL) integral for a function $y(x) \in L^1([x_0, T])$; as usual, L^1 is the set of Lebesgue integrable functions, the RL fractional integral of order $a > 0$ and origin at x_0 is defined as:

$$J_{x_0}^{\alpha} y(x) := \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-s)^{\alpha-1} y(s) \qquad (2.1)$$

Indeed, the particular case for the Riemann–Liouville integral (2.1) when $a = 0$, the left inverse of $J_{x_0}^{\alpha} y(x)$ is the Riemann–Liouville fractional derivative:

$$\hat{D}_{x_0}^{\alpha} y(x) := D^m J_{x_0}^{m-\alpha} y(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_{x_0}^{x} (x-s)^{m-\alpha-1} y(s) \qquad (2.2)$$

where $m = \lceil \alpha \rceil$ is the smallest integer greater or equal to α .

An alternative definition of the fractional derivative, obtained after interchanging differentiation and integration in Equation (2.2), is the so called Caputo derivative, which, for a sufficiently differentiable function, that is to say for $y^m \in A^m([x_0, T])$, where y^m is absolutely continuous given by:

$$D_{x_0}^{\alpha} y(x) := J_{x_0}^{m-\alpha} D^m y(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-s)^{m-\alpha-1} y^{(m)}(s) ds \qquad (2.3)$$

The left inverse of the Riemann–Liouville integral is $D_{x_0}^{\alpha} y(x)$, that is $D_{x_0}^{\alpha} J_{x_0}^{\alpha} y = y$, but not its right inverse, see [6]:

$$J_{x_0}^{\alpha} D_{x_0}^{\alpha} y = y(x) - T^{m-1}[y, x_0](x) \qquad (2.4)$$

where $T^{m-1}[y, x_0](x)$ is the Taylor polynomial of degree $m-1$ for the function $y(x)$ centered at x_0, that is:

$$T^{m-1}[y, x_0](x) = \sum_{k=0}^{m-1} \frac{(x-x_0)}{k!} y^k(x_0)$$

Now by deriving both sides of Equation (2.4) in the Riemann–Liouville, it is probable to observe that:

$$D_{x_0}^{\alpha} y(x) = \hat{D}_{x_0}^{\alpha} \left[y(x) - T^{m-1}[y, x_0](x) \right] \qquad (2.5)$$

Consequently, we have:

$$\hat{D}_{x_0}^{\alpha} y(x) = D_{x_0}^{\alpha} y(x) + \sum_{k=0}^{m-1} \frac{(x-x_0)^{k-\alpha}}{\Gamma(k+\alpha-1)} y^k(x_0) \qquad (2.6)$$

Observe that the above relationship it has special case when $0 < \alpha < 1$, so (2.6) becomes:

$$\hat{D}_{x_0}^{\alpha} y(x) = D_{x_0}^{\alpha} y(x) + \frac{(x-x_0)^{-\alpha}}{\Gamma(1-\alpha)} y(x_0)$$

The initial value problem for Fractional differential equation (or a system of FDEs) with Caputo's derivative can be formulated as:

$$D_{x_0}^{\alpha} y(x) = f(x, y(x)) \qquad (2.7)$$

$$y(x_0) = y_0, y'(x_0) = y_0^{(1)}, \ldots, y^{(m-1)}(x_0) = y_0^{(m-1)}$$

where $f(x, y(x))$ is assumed to be continuous and $y_0, y_0^{(1)}, \ldots, y_0^{(m-1)}$ are the values of the derivatives at x_0. The application to both sides of Equation (2.6) of the Riemann–Liouville integral $J_{x_0}^{\alpha}$, together with Equation (2.3),leads to the reformulation of the fractional differential equations in terms of the weakly-singular Volterra integral equation:

$$y(x) = T^{m-1}[y, x_0](x) + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-s)^{\alpha-1} f(s, y(s)) ds \qquad (2.8)$$

The integral Formula is used in the theoretical and numerical results and available for this class of Volterra integral equations in order to study and solve fractional differential equations, see [6]. The existence and uniqueness of solution to fractional order ordinary and delay differential equations discussed by Syed Abbas[20], and shown the existence of the solutions of the differential equations:

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = g(t, x(t))$$

$$x(0) = x_0; \quad 0 < \alpha < 1, \quad t \in [0,T]$$

and

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = f(t, x(t), x(t-\tau)); \quad t \in [0,T]$$

$$x(t) = \phi(t); \quad t \in [-\tau, 0]; \quad 0 < \alpha < 1$$

under suitable conditions on g, f and $\phi(t)$.

3. Numerical Methods

In1900, C. Runge and M. W. Kutta were developed the classical 4th order Runge-Kutta techniques. Then after that, this method took a major role in the study of iterative methods based on explicit and implicit, which applied to solve ordinary differential equations. The Runge-Kutta method is numerical method used to solve a system of ODEs with suitable initial conditions.In [21] introduced a general formula of Runge-Kutta method in order four with a free parameter. The authors constructed the modified Runge-Kutta method and showed that this method preserves the order of accuracy of the original one (see [8]).

Now, consider the initial value problem:

$$y'(x) = f(x, y(x)); \quad y(x_0) = y_0 \qquad (3.1)$$

Define h to be the time step size and $x_i = x_0 + ih$. So, we need some definitions:

Firstly, the formula for the fourth orders Runge-Kutta method for initial value problem (3.1) is given by:

$$\begin{aligned}
k_1 &= hf(x_i, y_i) \\
k_2 &= hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}) \\
k_3 &= hf(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}) \qquad (3.2) \\
k_4 &= hf(x_i + h, y_i + k_3) \\
y_{i+1} &= y_i + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}; \quad i = 0,1,2,....
\end{aligned}$$

Secondly, the formula for the modified Runge-Kutta method for initial value problem (3.1) is given by:

$$\begin{aligned}
k_1 &= hf(x_i, y_i) \\
k_2 &= hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}) \\
k_3 &= hf(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}) \qquad (3.3) \\
k_4 &= hf(x_i + h, y_i + k_3) \\
k_5 &= hf(x_i + \frac{3}{4}h, y_i + \frac{2}{32}(5k_1 + 7k_2 + 13k_3 - k_4))
\end{aligned}$$

Then an approximation to the solution of initial value problem is made using higher order
Runge-Kutta method of order 4:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_5); \qquad i = 0,1,2,....$$

The local truncation error at each step can be estimated using the following relation:

$$E_r = \frac{2h}{3}(-k_1 + 3k_2 + 3k_3 + 3k_4 - 8k_5)$$

Thirdly, the improvement version of classical Runge-Kutta method for IVP (3.1) which called Runge-Kutta Mersion method with the global error $O(h^4)$, it can be written as the form (see[9]):

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_4 + k_5); \qquad i = 0,1,2,....$$

where k_1, k_2, k_3, k_4, k_5 are given by:

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf(x_i + \frac{h}{3}, y_i + \frac{k_1}{2}) \qquad (3.4)$$

$$k_3 = hf(x_i + \frac{h}{3}, y_i + \frac{(k_1 + k_2)}{6})$$

$$k_4 = hf(x_i + \frac{h}{2}, y_i + \frac{(k_1 + k_2)}{8})$$

$$k_5 = hf(x_i + h, y_i + \frac{1}{2}(k_1 - 3k_3 + 4k_4))$$

with the local truncation error at each step can be using by the following formula :

$$E_r = \frac{1}{3}(2k_1 - 9k_3 + 8k_4 - k_5)$$

4. Main Results

In this section, we present a study on the numerical solution of initial value problem for second order nonlinear equations of Bernoulli type with fractional derivative [22], which can be written in the form:

$$P(x)D^2 y + R(x)D^\alpha y + Q(x)Dy + S(x)y = m\,P(x)\frac{y'^2}{y} + \frac{R(x)}{\Gamma(1-\alpha)x^\alpha} y + f(x)y^m \qquad (4.1)$$

subject to following initial condition: $y(a) = y_0, D^\alpha y(a) = D^\alpha y_0, D^{\alpha+1} y(a) = D^{\alpha+1} y_0, ...,$
$D^{2-\alpha} y(a) = D^{2-\alpha} y_0, D^2 y(a) = D^2 y_0$ where: $P(x) \neq 0, Q(x) \neq 0, m \geq 2$ and also y_0, y'_0 are not equal to zero, where $1 \leq \alpha \leq 2$.

To find solution for this type of differential equations, we shall reduce the Bernoulli's equation (4.1) to the linear equation by the transformation $u = y^{1-m}$, and hence (4.1) will becomes as (see[22]):

$$\frac{1}{1-m}\left(P(x)\frac{d^2u}{dx^2}+R(x)D^{\alpha}u+Q(x)\frac{du}{dx}\right)+S(x)u=f(x) \qquad (4.2)$$

subject to the initial conditions

$$u(a)=y^{1-m}(a),\ D^{\alpha}u(a)=D^{\alpha}\left(y^{1-m}(a)\right), D^{\alpha+1}u(a)=D^{\alpha+1}\left(y^{1-m}(a)\right), ...,$$
$$D^{2-\alpha}u(a)=D^{2-\alpha}\left(y^{1-m}(a)\right), D^{2}u(a)=D^{2}\left(y^{1-m}(a)\right) \qquad (4.3)$$

To find numerical solutions for initial value problem (4.2),(4,3) by the 4th Runge-Kutta method, modified Runge-Kutta method and Runge-Kutta Mersion method , we firstly rewrite this problem as system of ordinary differential equations of fractional derivatives with initial values, as follow: Let $u = u_1$ hence we have:

$$\begin{aligned}
&D^{\alpha}u_1 = u_2\ , && u_1(a)=u(a)\\
&D^{\alpha}u_2 = u_3\ , && u_2(a)=D^{\alpha}u(a)\\
&D^{\alpha}u_3 = u_4\ , && u_3(a)=D^{\alpha+1}u(a) \qquad (4.3)\\
&\vdots && \vdots\\
&D^{\alpha}u_N = \frac{1-m}{P(x)}\left(f(x)-S(x)u_1\right)-\frac{1}{P(x)}\left(R(x)u_2+Q(x)\frac{du_1}{dx}\right), && u_N(a)=D^2u(a)
\end{aligned}$$

Consequently, we will apply the fourth order Runge-Kutta method for this system of fractional differential equations, as the follows:

$$\begin{aligned}
k_{j1} &= hf_j(x_i, u_{1,i}, u_{2,i}, u_{3,i}, ..., u_{N,i})\\
k_{j2} &= hf_j(x_i+\frac{h}{2}, u_{1,i}+\frac{k_{11}}{2}, u_{2,i}+\frac{k_{21}}{2}, u_{3,i}+\frac{k_{31}}{2}, ..., u_{N,i}+\frac{k_{N1}}{2}) \qquad (4.4)\\
k_{j3} &= hf_j(x_i+\frac{h}{2}, u_{1,i}+\frac{k_{12}}{2}, u_{2,i}+\frac{k_{22}}{2}, u_{3,i}+\frac{k_{32}}{2}, ..., u_{N,i}+\frac{k_{N2}}{2})\\
k_{j4} &= hf_j(x_i+h, u_{1,i}+k_{13}, u_{2,i}+k_{23}, u_{3,i}+k_{33}, ..., u_{N,i}+k_{N3})
\end{aligned}$$

Consequently, we compensate $k_{j1}, k_{j2}, k_{j3}, k_{j4};\ j=1,2,...,N$ in the following iterations:

$$u_{1,i+1}=u_{1,i}+\frac{1}{6}\left(k_{11}+2k_{12}+2k_{13}+k_{14}\right);\quad i=0,1,2,...$$
$$u_{2,i+1}=u_{2,i}+\frac{1}{6}\left(k_{21}+2k_{22}+2k_{23}+k_{24}\right)$$
$$u_{3,i+1}=u_{3,i}+\frac{1}{6}\left(k_{31}+2k_{32}+2k_{33}+k_{34}\right) \tag{4.5}$$
$$\vdots$$
$$u_{N,i+1}=u_{N,i}+\frac{1}{6}\left(k_{N1}+2k_{N2}+2k_{N3}+k_{N4}\right)$$

Secondly, we find approximate solution for initial value problem (4.2) , (4.3) by applying the modified Runge-Kutta method is given by:

$$k_{j5}=hf_j\,(x_i+\frac{3}{4}h,u_{1,i}+\Phi_1,u_{2,i}+\Phi_2,u_{3,i}+\Phi_3,...,u_{N,i}+\Phi_N) \tag{4.6}$$

where:

$$\Phi_j=\frac{2}{32}\left(5k_{j1}+7k_{j2}+13k_{j3}-k_{j4}\right),\quad j=1,2,...,N \tag{4.7}$$

Then an approximation to the solution of initial value problem is made using higher order
Runge-Kutta method of order four:

$$u_{1,i+1}=u_{1,i}+\frac{1}{6}\left(k_{11}+2k_{12}+2k_{13}+k_{15}\right);\quad i=0,1,2,...$$
$$u_{2,i+1}=u_{2,i}+\frac{1}{6}\left(k_{21}+2k_{22}+2k_{23}+k_{25}\right) \tag{4.8}$$
$$u_{3,i+1}=u_{3,i}+\frac{1}{6}\left(k_{31}+2k_{32}+2k_{33}+k_{35}\right)$$
$$\vdots$$
$$u_{N,i+1}=u_{N,i}+\frac{1}{6}\left(k_{N1}+2k_{N2}+2k_{N3}+k_{N5}\right)$$

Thirdly, the Runge-Kutta Mersion method for system of fractional differential equations (4.3), it can be written as the form:

$$u_{j,i+1}=u_{j,i}+\frac{1}{6}\left(k_{j1}+4k_{j4}+k_{j5}\right);\quad i=0,1,2,...\ ;\ j=1,2,...,N \tag{4.9}$$

where: $k_{j1},k_{j2},k_{j3},k_{j4},k_{j5};\ j=1,2,...,N$ are taken the formula:

$$k_{j1} = hf_j(x_i, u_{1,i}, u_{2,i}, u_{3,i}, ..., u_{N,i}) \; ;$$

$$k_{j2} = hf_j(x_i + \frac{h}{3}, u_{1,i} + \frac{k_{11}}{2}, u_{2,i} + \frac{k_{21}}{2}, u_{3,i} + \frac{k_{31}}{2}, ..., u_{N,i} + \frac{k_{N1}}{2}) \quad (4.10)$$

$$k_{j3} = hf_j(x_i + \frac{h}{3}, u_{1,i} + \frac{k_{11}+k_{12}}{6}, u_{2,i} + \frac{k_{21}+k_{22}}{6}, u_{3,i} + \frac{k_{31}+k_{32}}{6}, ..., u_{N,i} + \frac{k_{N1}+k_{N2}}{6})$$

$$k_{j4} = hf_j(x_i + \frac{h}{2}, u_{1,i} + \frac{k_{11}+k_{13}}{8}, u_{2,i} + \frac{k_{21}+k_{23}}{8}, u_{3,i} + \frac{k_{31}+k_{33}}{8}, ..., u_{N,i} + \frac{k_{N1}+k_{N3}}{8})$$

$$k_{j5} = hf_j(x_i + h, u_{1,i} + \bar{\Phi}_1, u_{2,i} + \bar{\Phi}_2, u_{3,i} + \bar{\Phi}_3, ..., u_{N,i} + \bar{\Phi}_N)$$

where:

$$\bar{\Phi}_j = \frac{1}{2}\left(k_{j1} - 3k_{j3} + 4k_{j4}\right), \quad j = 1,2,...,N \quad (4.11)$$

5. Illustrative Examples

In this section, we are applying the methods presented in Section 4 to solve the following examples and verify that these numerical methods converge to the exact solution quickly or not, and calculate the numerical solutions and errors.

Example 1. Consider the second order of nonlinear Bernuolli equation with fractional derivative:

$$D^2 y - D^{\frac{1}{2}} y - Dy = 2\frac{y'^2}{y} - \frac{y}{\sqrt{\pi x}} + \left(\frac{1}{\sqrt{\pi x}} + 2x + \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}} - 2\right) y^2 \quad (5.1)$$

with initial conditions: $y(1) = \frac{1}{2}, D^{\frac{1}{2}} y(1) = \frac{-5}{12\sqrt{\pi}}, D^{\frac{3}{2}} y(1) = \frac{-9}{8\sqrt{\pi}}, y'(1) = \frac{-1}{2}$

The exact solution is: $y = \frac{1}{x^2+1}$. To finds solution for nonlinear differential equation (5.1), first, we shall reduce the Bernoulli equation to the linear equation by the transformation $u = y^{-1}$, and hence the equation will become to:

$$\frac{d^2 u}{dx^2} - \frac{du}{dx} - D^{\frac{1}{2}} u = 2 - \frac{1}{\sqrt{\pi x}} - 2x - \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}} \quad (5.2)$$

subject to the initial conditions: $u(1)=2, D^{\frac{1}{2}} u(1)=\frac{11}{3\sqrt{\pi}}, D^{\frac{3}{2}} u(1)=\frac{7}{2\sqrt{\pi}}, u'(1)=2$

Now; let $u = u_1$, and after that we applying Runge-Kutta method (4.5), modified Runge-Kutta method (4.8) and Runge kutta Merson method (4.9) for the following system:

$$D^{\frac{1}{2}}u_1 = u_2 \,, \quad u_{1,0}(1) = 2$$

$$D^{\frac{1}{2}}u_2 = u_3 \,, \quad u_{2,0}(1) = \frac{11}{3\sqrt{\pi}}$$

$$D^{\frac{1}{2}}u_3 = u_4 \,, \quad u_{3,0}(1) = 2, \; u_{4,0}(1) = \frac{7}{2\sqrt{\pi}} \tag{5.3}$$

$$D^{\frac{1}{2}}u_4 = 2 - \frac{1}{\sqrt{\pi x}} - 2x - \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}} + u_2 + u_3$$

Table 1: Comparison of the exact solution and numerical solutions by 4th order Runge- Kutta method modified Runge kutta method and Runge kutta Mersion method, which is displayed in Fig. 1 for the step size $h = 0.1$.

x_i	*Exact u*	*RK*	*MRK*	*RKM*	*Error with respect to RK*	*Error with respect to MRK*	*Error with respect to RKM*
1.0	2.00	2.00	2.00	2.00	0.00	0.00	0.00
1.1	2.21	2.21721	2.18996	2.18267	0.00720696	0.0200392	0.0273263
1.2	2.44	2.45651	2.39785	2.3816	0.0165054	0.04221529	0.0584044
1.3	2.69	2.72017	2.6255	2.59831	0.0301719	0.0645013	0.0916943
1.4	2.96	3.01069	2.87491	2.83445	0.0550689	0.0850928	0.125548
1.5	3.25	3.33075	3.14822	3.09179	0.0807521	0.1017788	0.158213
1.6	3.56	3.68328	3.44776	3.37217	0.123281	0.11224	0.187831
1.7	3.89	4.07143	3.77602	3.67756	0.181433	0.113982	0.212443
1.8	4.24	4.49862	4.13569	4.01002	0.258623	0.104308	0.229982
1.9	4.61	4.96855	4.5297	4.37173	0.3585547	0.0803039	0.238272
2.0	5.00	5.48522	4.9612	4.76498	0.485218	0.0388009	0.235024

Table 1

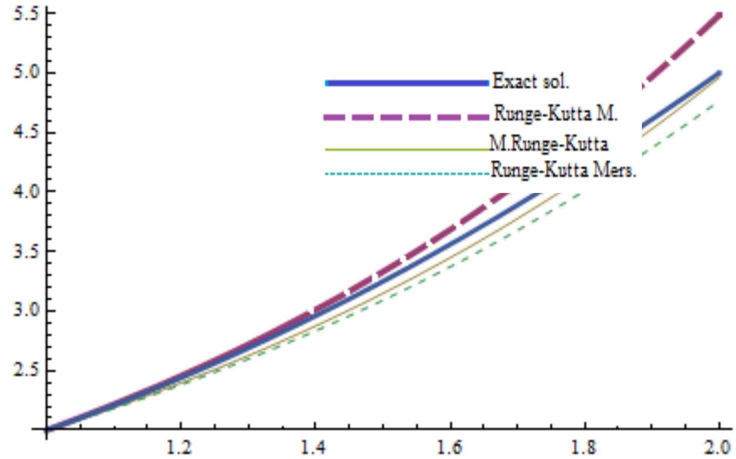

Fig. 1. Comparing Exact solution of $u(x)$ with Runge-Kutta techniquesfor $\alpha = \frac{1}{2}$

Example 2. Consider the second order of nonlinear Bernoulli equation with fractional derivative:

$$D^2y - x\,D^{\frac{3}{2}}y + Dy = 2\frac{y'^2}{y} + \frac{y}{2\sqrt{\pi x}} + \left(\frac{4}{\sqrt{\pi}}x^{\frac{3}{2}} + \frac{8}{\sqrt{\pi}}x^{\frac{5}{2}} - 2 - 8x - 3x^2\right)y^2 \tag{5.4}$$

The exact solution for this problem is given by: $y = \frac{1}{x^2 + x^3}$, with initial conditions: $y(1) = \frac{1}{2}$, $D^{\frac{1}{2}} y(1) = \frac{-29}{30\sqrt{\pi}}$, $y'(1) = \frac{-5}{4}$, $D^{\frac{3}{2}} y(1) = \frac{-13}{4\sqrt{\pi}}$, $y''(1) = \frac{17}{4}$.

To find approximate solution for nonlinearfractional differential equation, we shall reduce this problem to the linear equation firstly and we applying transformation $u = y^{-1}$, hence the equation (5.4) will becomes:

$$\frac{d^2u}{dx^2} - x\, D^{\frac{3}{2}} u + \frac{du}{dx} = 2 + 8x + 3x^2 - \frac{4}{\sqrt{\pi}} x^{\frac{3}{2}} - \frac{8}{\sqrt{\pi}} x^{\frac{5}{2}} \qquad (5.5)$$

subject to the initial conditions: $u(1) = 2$, $D^{\frac{1}{2}} u(1) = \frac{88}{15\sqrt{\pi}}$, $u'(1) = 5$, $D^{\frac{3}{2}} u(1) = \frac{12}{\sqrt{\pi}}$.

Now, let $u = u_1$, therefore we applying 4th order Runge-Kutta method (4.5), modified Runge Kutta method (4.8) and Runge Kutta Mersion method (4.9) for the following system:

$$D^{\frac{1}{2}} u_1 = u_2 , \quad D^{\frac{1}{2}} u_2 = u_3 , \quad D^{\frac{1}{2}} u_3 = u_4 ,$$

$$D^{\frac{1}{2}} u_4 = 2 + 8x + 3x^2 - \frac{4}{\sqrt{\pi}} x^{\frac{3}{2}} - \frac{8}{\sqrt{\pi}} x^{\frac{5}{2}} - u_3 + x\, u_4 \qquad (5.6)$$

$$u_{1,0}(1) = 2, \quad u_{2,0}(1) = \frac{88}{15\sqrt{\pi}}, \quad u_{3,0}(1) = 5, \quad u_{4,0}(1) = \frac{12}{\sqrt{\pi}}$$

Table 2: Comparison of the exact solution and numerical solutions by 4th order Runge- Kutta method modified Runge kutta method and Runge kutta Mersion method, which is displayed in **Fig. 2** for the step size $h = 0.1$.

x_i	*Exact u*	*RK*	*MRK*	*RKM*	*Error with respect to RK*	*Error with respect to MRK*	*Error with respect to RKM*
1.0	2.00	2.00	2.00	2.00	0.00	0.00	0.00
1.1	2.541	2.35715	2.33661	2.31192	0.183847	0.204393	0.229082
1.2	3.168	2.77156	2.73404	2.67364	0.396439	0.433956	0.49436
1.3	3.887	3.25129	3.20489	3.09392	0.63571	0.6822108	0.793084
1.4	4.704	3.80536	3.76472	3.58327	0.898644	0.939281	1.12073
1.5	5.625	4.44382	4.43289	4.15442	1.18118	1.19211	1.47058
1.6	6.656	5.17786	5.23363	4.82274	1.47814	1.42237	1.83326
1.7	7.803	6.01991	6.19751	5.60697	1.78309	1.60549	2.19603
1.8	9.072	6.98371	7.36352	6.53007	2.0889	1.70848	2.54193
1.9	10.469	8.08451	8.78211	7.62046	2.38449	1.68689	2.84854
2.0	12.0	9.33915	10.5199	8.91376	2.66085	1.48012	3.08624

Table 2

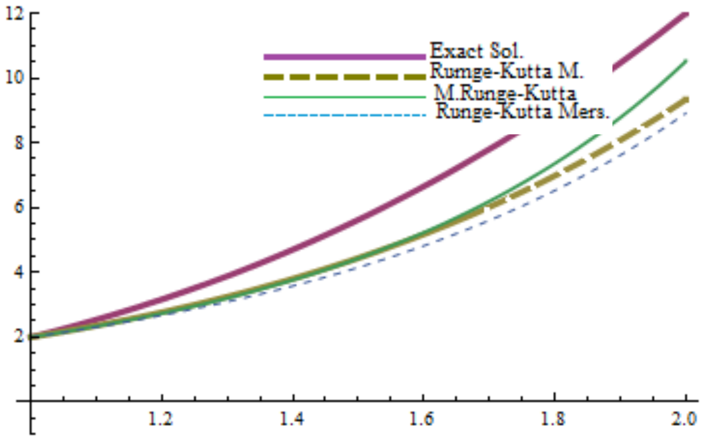

Fig. 2. Comparing Exact solution of $u(x)$ with Runge-Kutta techniquesfor $\alpha = \frac{3}{2}$

Example 3. Consider the second order of nonlinear Bernuolli equation with fractional derivative:

$$D^2 y + x\,Dy + \frac{\sqrt{\pi}}{2} D^{\frac{1}{2}} y + \frac{1}{2} y = 3\frac{y'^2}{y} + \frac{y}{2\sqrt{x}} + \left(\frac{\sqrt{x}}{2} + \frac{3\pi}{12} x - \frac{1}{8\sqrt{x}} - \frac{5}{12} x^{\frac{3}{2}} - \frac{x^2}{2}\right) y^3 \qquad (5.7)$$

with initial conditions: $y(1) = 1, y'(1) = \frac{-1}{4}, y''(1) = \frac{-7}{16}$.

The exact solution is: $y = \frac{1}{\sqrt{x^2 - x^{\frac{3}{2}} - x + 2}}$. Therefore, to find approximate solution for nonlinear differential equation (5.7), we shall reduce the Bernoulli's equation to the linear equation by the transformation $u = y^{-2}$, hence the equation will becomes:

$$\frac{d^2u}{dx^2} + x\frac{du}{dx} + \frac{\sqrt{\pi}}{2} D^{\frac{1}{2}} u - u = \frac{1}{4\sqrt{x}} - \sqrt{x} - \frac{3\pi}{6} x + \frac{5}{6} x^{\frac{3}{2}} + x^2 \qquad (5.8)$$

subject to the initial conditions: $u(1)=1, D^{1/2}u(1)=\frac{8}{3\sqrt{\pi}}-\frac{3\sqrt{\pi}}{4}$, $u'(1)=\frac{-1}{2}$, $D^{3/2}u(1)=\frac{2}{\sqrt{\pi}}-\frac{3\sqrt{\pi}}{4}$.

Consequently, we applying Rune Kutta method (*RK*), modified Rune Kutta (*MRK*) and Runge kutta Merson method (*RKM*) for the following system:

$$D^{\frac{1}{2}}u_1 = u_2\ ,\quad u_{1,0}(1) = 1$$

$$D^{\frac{1}{2}}u_2 = u_3\ ,\quad u_{2,0}(1) = \frac{8}{3\sqrt{p}} - \frac{3\sqrt{p}}{4} \tag{5.9}$$

$$D^{\frac{1}{2}}u_3 = u_4\ ,\quad u_{3,0}(1) = \frac{-1}{2}\quad ,\quad u_{4,0}(1) = \frac{2}{\sqrt{p}} - \frac{3\sqrt{p}}{4}$$

$$D^{\frac{1}{2}}u_4 = \frac{1}{4\sqrt{x}} - \sqrt{x} - \frac{3px}{6} + \frac{5x^{\frac{3}{2}}}{6} + x^2 + u_1 - \frac{\sqrt{p}}{2}u_2 - xu_3$$

Table 3: Comparison of the exact solution and numerical solutions by 4th order Runge Kutta method modified Runge kutta method and Runge kutta Mersion method, , which is displayed in **Fig. 3** for the step size $h = 0.1$.

x_i	*Exact* u	*RK*	*MRK*	*RKM*	*Error with respect to RK*	*Error with respect to MRK*	*Error with respect to RKM*
1.0	1.00	1.00	1.00	1.00	0.00	0.00	0.00
1.1	0.95631	1.01499	1.01457	1.01584	0.058678	0.0582575	0.0595341
1.2	0.925466	1.02486	1.02399	1.02658	0.0993896	0.098529	0.101111
1.3	0.907772	1.02962	1.02834	1.03214	0.121852	0.120568	0.124366
1.4	0.903498	1.0295	1.02785	1.03263	0.126001	0.124354	0.129137
1.5	0.912883	1.0249	1.023	1.02837	0.112014	0.110113	0.115491
1.6	0.936142	1.01647	1.01448	1.01989	0.0803318	0.0783394	0.0837473
1.7	0.973471	1.00515	1.00329	1.00797	0.0316812	0.0298172	0.0345015
1.8	1.02505	0.992138	0.990681	0.998433	0.0329083	0.0343654	0.0313512
1.9	1.09103	0.978941	0.978227	0.978433	0.11209	0.112804	0.112598
2.0	1.17157	0.96738	0.9678	0.963878	0.204193	0.203773	0.207695

Table 3

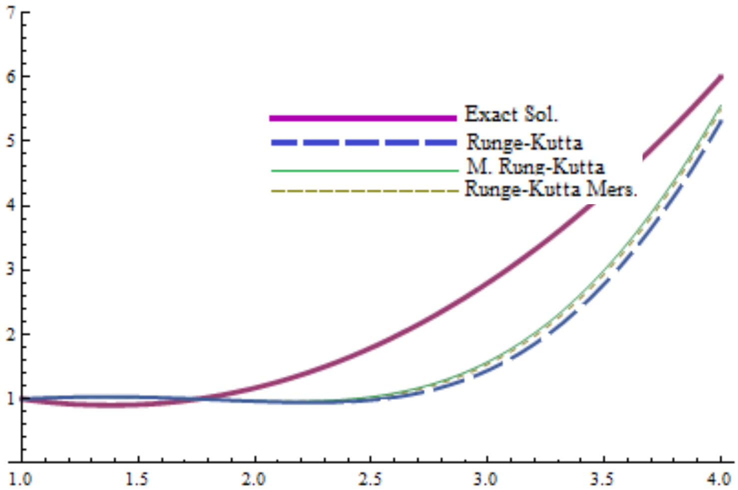

Fig. 3. Comparing Exact solution of $u(x)$ with Runge-Kutta techniquesfor $\alpha = \frac{1}{2}$

Example 4. Consider the second order of nonlinear Bernuolli equation with fractional derivative:

$$D^2y + x^{\frac{2}{3}} D^{\frac{4}{3}}y - Dy = 2\frac{y'^2}{y} + \frac{y}{\Gamma(\frac{-1}{3})x^{\frac{4}{3}}} + \left(-2x + x^2 - \frac{2}{\Gamma(\frac{8}{3})}x^{\frac{7}{3}}\right)y^2 \qquad (5.10)$$

with initial conditions: $y(1)=3, y'(1)=-9, y''(1)=36$. To find the approximate solution for nonlinear differential equation (5.10), we shall reduce the Bernoulli's equation to the linear equation by the transformation $u = y^{-1}$, hence the equation will becomes:

$$\frac{d^2u}{dx^2} + x^{\frac{2}{3}} D^{\frac{4}{3}}u - \frac{du}{dx} = 2x - x^2 + \frac{2}{\Gamma(\frac{8}{3})}x^{\frac{7}{3}} \qquad (5.11)$$

subject to the initial condition: $u(1)=\frac{1}{3}, D^{\frac{1}{3}}u(1)=\frac{2}{G(\frac{11}{3})}, D^{\frac{2}{3}}u(1)=\frac{2}{G(\frac{10}{3})}, Du(1)=1,$ $D^{\frac{4}{3}}u(1)=\frac{2}{G(\frac{8}{3})}, D^{\frac{5}{3}}u(1)=\frac{2}{G(\frac{7}{3})}$. Consequently, we applying the 4th order Runge-Kutta method (*RK*), modified Rune Kutta (*MRK*) and Runge kutta Mersion method (*RKM*).

$$D^{\frac{1}{3}}u_1 = u_2, \quad D^{\frac{1}{3}}u_2 = u_3, \quad D^{\frac{1}{3}}u_3 = u_4,$$
$$D^{\frac{1}{3}}u_4 = u_5, \quad D^{\frac{1}{3}}u_5 = u_6, \qquad (5.12)$$
$$D^{\frac{1}{3}}u_6 = 2x - x^2 + \frac{2}{G(\frac{8}{3})}x^{\frac{7}{3}} - x^{\frac{2}{3}}u_5 + u_4$$
$$u_{1,0}(1)=\frac{1}{3}, u_{2,0}(1)=\frac{2}{G(\frac{11}{3})}, u_{3,0}(1)=\frac{2}{G(\frac{10}{3})}, u_{4,0}(1)=\frac{2}{G(\frac{8}{3})}, u_{5,0}(1)=\frac{2}{G(\frac{7}{3})}$$

Table 4: Comparison of the exact solution and numerical solutions by Rune Kutta method modified Runge kutta method and Runge kutta Mersion method, which is displayed in **Fig.4**.

x_i	*Exact u*	*RK*	*MRK*	*RKM*	*Error with respect to RK*	*Error with respect to MRK*	*Error with respect to RKM*
1.0	0.333333	0.333333	0.333333	0.333333	0.00	0.00	0.00
1.1	0.443667	0.38687	0.378273	0.385637	0.056797	0.0653932	0.0580301
1.2	0.576	0.447926	0.428647	0.444529	0.128074	0.147353	0.131471
1.3	0.732333	0.516792	0.484695	0.510843	0.215541	0.247638	0.22149
1.4	0.914667	0.593811	0.546707	0.585531	0.320856	0.36796	0.329135
1.5	1.125	0.679386	0.615025	0.669688	0.445614	0.509975	0.455312
1.6	1.36533	0.773995	0.690061	0.764568	0.591338	0.675272	0.600766
1.7	1.63767	0.878199	0.772306	0.871611	0.759468	0.86536	0.766055
1.8	1.944	0.992658	0.862346	0.992471	0.951342	1.08165	0.9951529
1.9	2.28633	1.11815	0.960881	1.12904	1.16818	1.32545	1.15729
2.00	2.66667	1.25558	1.06874	1.2835	1.41109	1.59792	1.38317

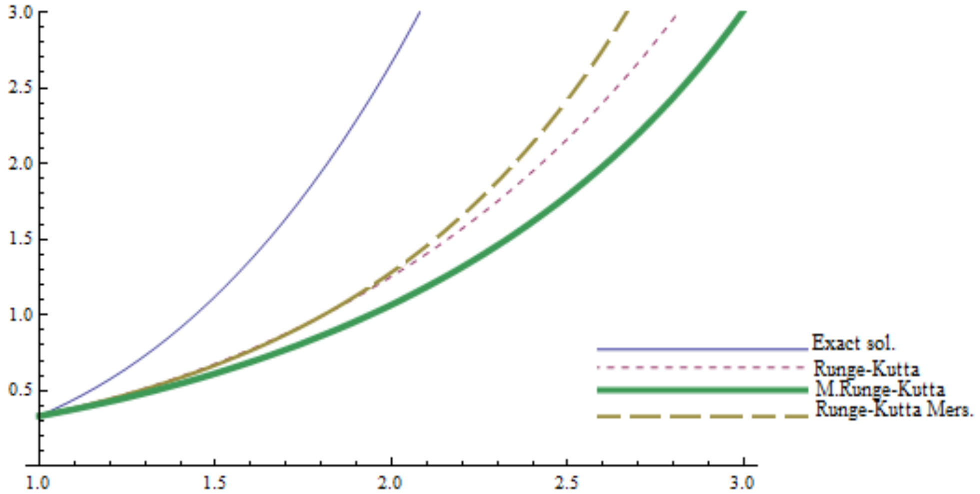

Fig. 4. Comparing Exact solution of $u(x)$ with Runge-Kutta techniquesfor $\alpha = \frac{4}{2}$

Example 5. Consider the second order of nonlinear Bernuolli equation with fractional derivative:

$$D^2 y + 2Dy - 3D^{\frac{1}{2}} y = 4\frac{y'^2}{y} - \frac{3y}{\sqrt{\pi x}} + \left(\frac{2}{\sqrt{\pi x}} - 4\right) Exp(-3x) y^4 \qquad (5.13)$$

with initial conditions: $y(1) = D^{\frac{1}{2}} y(1) = y'(1) = D^{\frac{3}{2}} y(1) = e$, the exact solution of this equation is: $y = Exp(x)$.To find approximate solution for nonlinear fractional differential equation (5.13), we shall reduce Bernoulli's equation to a linear equation by transformation$u = y^{-3}$, hence the previous equation will becomes:

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} - 3D^{\frac{1}{2}} u = \left(4 - \frac{2}{\sqrt{\pi x}}\right) Exp(-3x) \qquad (5.14)$$

subject to the initial conditions: $u(1) = e^{-3}$, $D^{\frac{1}{2}} u(1) = (\frac{1}{\sqrt{\pi}} - \sqrt{3})e^{-3}$, $u'(1) = -3e^{-3}$

$D^{\frac{3}{2}} u(1) = (\frac{-1}{2\sqrt{\pi}} - 3\sqrt{3})e^{-3}$. Consequently, we applying 4th order Runge-Kutta (4.4), modified Runge Kutta (4.8) and Runge Kutta Mersion (4.9) for the following system:

$D^{\frac{1}{2}} u_1 = u_2$, $D^{\frac{1}{2}} u_2 = u_3$, $D^{\frac{1}{2}} u_3 = u_4$,

$$D^{\frac{1}{2}} u_4 = (4 - \frac{x}{\sqrt{px}}) Exp(-3x) + 3u_2 - 2u_3 \qquad (5.15)$$

$$u_{1,0}(1) = e^{-3},\ u_{2,0}(1) = (\frac{1}{\sqrt{p}} - \sqrt{3})e^{-3},\ u_{3,0}(1) = -3e^{-3},\ u_{4,0}(1) = (\frac{-1}{2\sqrt{p}} - \sqrt{3})e^{-3}$$

Table 5: Comparison of the exact solution and approximate solutions by 4th order Runge- Kutta method, modified Runge-Kutta method and Runge-Kutta Mersion method, which is displayed in **Fig. 5** for the step size $h = 0.1$.

x_i	Exact u	RK	MRK	RKM	Error with respect to RK	Error with respect to MRK	Error with respect to RKM
1.0	0.0497871	0.0497871	0.0497871	0.0497871	0.00	0.00	0.00
1.1	0.0368832	0.0431816	0.0444443	0.0446999	0.00629843	0.00756108	0.00781672
1.2	0.0273237	0.034826	0.0379366	0.0385131	0.00750228	0.0106129	0.0111894
1.3	0.0202419	0.0244868	0.0301269	0.0310797	0.0042486	0.00988504	0.0108378
1.4	0.0149956	0.0119516	0.0208902	0.0222659	0.00304395	0.00589459	0.00727029
1.5	0.011109	-0.0029749	0.0101108	0.011949	0.0140839	0.00998148	0.0008401
1.6	0.00822975	-0.0204763	-0.00231857	0.0000163468	0.0287061	0.0105483	0.0082134
1.7	0.00609675	-0.0407294	-0.0164998	-0.0136375	0.0468261	0.0225966	0.0197343
1.8	0.00451658	-0.0639093	-0.0325308	-0.0291122	0.0684259	0.0370474	0.0336288
1.9	0.00334597	-0.0901959	-0.0505078	- 0.0465034	0.0935415	0.0538538	0.0498494
2.0	0.00247875	-0.119779	-0.0705273	-0.0659053	0.122258	0.073006	0.0683841

Table 5

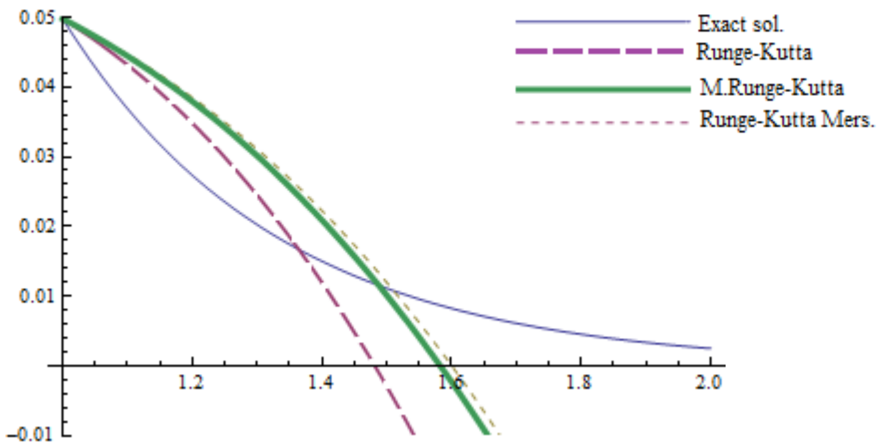

Fig. 5. Comparing Exact solution of $u(x)$ with Runge-Kutta techniques for $\alpha = \frac{1}{2}$.

6. Conclusion

The aim of the present work is to find the approximate solution for the fractional differential equations of the Bernoulli Type. The problem has been reduced to solving a system of fractional differential equations under initial conditions. The desired approximate solution can be determined by solving the resulting system of linear fractional equations, which can be effectively computed using efficient and accurate methods. Finally, we applied the results on the some illustrative examples to show applicability of these techniques and we hoped that it would give us satisfactory results, but we didn't get the required.

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