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Complete polynmial vector fileds in Euclidean ball

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Abstract.
This work is on complete polynomial vector field in the unit ball of a Euclidean space by Theorem 2-4
Keywords. Polynomial vector field, complete vector filed.

1.Introduction

Given any subset K in R^N the set of all real N-ruples, a mapping $v: R^N \to R^N$ is said to be a complete vector field in K if for every point $k_0 \in K$ there exists a curve $x: R \to K$ such that $x(0) = k_0$ and $\frac{dx(t)}{dt} = v(x(t))$ for all reals $t \in R$. By definition $v: R^N \to K^N$ is a polynomial vector filed if $v(x) = (P_1(x), \dots, P_N(x))$ for any $x \in R^N$ with suitable polynomials $P_1, \dots, P_N: R^N \to R$ of N variables (that is each P_i is a finite linear combination of functions of the form $x_N^{mN} \dots x_N^{mN}$ with non-negative integers $x_j: (\xi_1, \dots, \xi_N) \to \xi_j$ denotes the j-th canonical function on R^N). By writing $\langle(\xi_1, \dots, \xi_N), (\eta_1, \dots, \eta_N)\rangle = \sum_{i=1}^{N} \xi_i \eta_i$ for the inner product in R^N, it is easy to see [2] that a polynomial (or even smooth) vector filed is complete in the unit ball B := $(\langle x, x\rangle < 1)$ iff it is complete in the sphere $S := (\langle x, x\rangle = 1)$. Furthermore, v is complete in S iff it is orthogonal to the radius vector on S i.e if $\langle v(x), x\rangle = 0$ for all $x \in S$.

In 2001 L.L Stacho [1] described the real polynomial vector fields of the unit disc D of the space C of complex numbers. He has shown that a real polynomial vector field $P: C \to C$
is complete in D iff P is a finite real linear combination of the function $iz, \gamma^{\bar{z}^m} - \bar{\gamma} z^{m+2}$ $(\gamma \in C, m = 0,1,2, \dots)$ and $(1 - |z|^2)Q$ where Q is any real polynomial $C \to C$. In this paper the complete polynomial vector fields of the Euclidean unit ball B (or equivalently the unit Euclidean sphere S) of R^N. We show that $P: R^N \to R^N$ is a complete polynomial vector filed in B if and only if $P(x) = R(x) - \langle R(x), x\rangle + (1 - \langle x, x\rangle)Q(x)$ for some polynomial vector fields Q, $R: R^N \to R^N$. Our result not only generalizes the result [1] on D, but it even simplifies it by showing that the the complete polynomial vector fields on the unit disc of C have the form $(ip(z)z + q(z)(1 - |z|^2))$ where $p, q: C \to R$ are any real polynomials. In our previous work [3] we represented complete vector fields on a simplex as polynomial combinations of finitely many basic complete vector fields. This idea motivated the formulation of our main result Theorem 2.3 and the Theorem 2.4.
Main result
2.1. Lemma. Let $f: R^N \to R$ be a polynomial such that $f(x) = 0$ for $x \in S$ where $S = (\langle x, x\rangle = 1)$. Then there exists a polynomial $Q: R^N \to R$, such that $f(x) = (1 - \langle x, x\rangle)Q(x)$.

Proof. Let $g: B \to R$ be function on the unit ball $B: (\langle x,x \rangle < 1)$, defined by $g(x) = \frac{f(x)}{(1-\langle x,x \rangle)}$. The function g is analytic, since it is the quotient of two polynomials.

Thus $g(x) = \sum_{k=0}^{\infty} g_k(x)$ where g_k are k-homogeneous polynomial on R^N. We have $f(\pm e) = 0$ whenever $\langle e,e \rangle = 1$, there exists a polynomial $P_e: R \to R$ of degree $\leq degf - 2$ such that $(1-t^2)P_e(t) = f(te)$. It follows that, for every fixed unit vector $e \in R^N$, $g(te) = \frac{f(te)}{(1-t^2)} = P_e(t) \sum_{k=0}^{degf-2} \alpha_k(e) t^k$ with suitable constants $\alpha_0(e), \ldots, \alpha_{deg-2}(e) \in R$. Thus $g = \sum_{k=0}^{degf-2} g_k$ is a polynomial. This completes the proof.

2.2. Remark. In classical algebraic geometry [4] an analogous result is known for irreducible sets is K^N where K is an algebraically closed field. However, we do not know any reference for the simple case of the lemma.

2.3 Theorem. Let $P: R^N \to R^N$ be a polynomial vector field. Then P is complete in the ball $B = (\langle x,x \rangle - 1)$(or equivalently in the sphere $S = (\langle x,x \rangle - 1)$) if and only if

$[P(x) = R(x) - \langle R(x), x \rangle x + (1 - \langle x,x \rangle) Q(x)]$.

See the paper [1] for the proof

2.4. Theorem. Let Ψ bounded continuously differentiable, $0 < \Psi \leq 1$ and $W: B(R^N) \to R^N$ is a complete vector field then ΨW is complete vector field.

proof. Let w(t) = $\int_{t=0}^{t} \psi(x_t)\, dt$ and $x_0 \in R^n$, x_t : $\frac{d}{dt}\big(x(t)\big) = w\big(x(t)\big)$, $t \in R$.

X(t) is well defined because W is a complete vector field

$y_t = x_{w(t)}$; $y_0 = x_{w(0)} = x_0$

$y'(t) = \frac{d}{dt} y(t) = \Psi(y(t))\, w(y(t))$

$y(t) = x(w(t))$; $x(0) = x_0$

$x'(t) = w(x(t))$

$y'(t) = \frac{d}{dt} y(t) = x'(w(t))w'(t) = w(x(w(t))).w'(t) = w(y(t)).w'(t)$.

$\Psi(y(t)) = w'(t)$; $w(0) = 0$

$\Psi(x(w(t))) = w'(t)$

$\varphi = \Psi o x$ Bounded By assumption (also C^1 _ smooth)

$w'(t) = \varphi(w(t))$; $w(0) = 0$

$$\frac{w'(t)}{\varphi(w(t))} = 1 \quad , \qquad \Phi(s) = \int_{\tau=0}^{s} \frac{d\tau}{\varphi(\tau)}$$

$\frac{d}{dt}\Phi\big(w(t)\big) = 1$; $\Phi(0) = 0$ monotonic increasing strictly imply that Φ^{-1} exists $\Phi(w(t)) = t$ then $w(t) = \Phi^{-1}(t)$ ■

2.5. Theorem. Let $V_k: x \to e_k - \langle e_k, x \rangle x$ where $k = 1,2,3,\ldots,N$.

Then every complete polynomial vector field on the sphere $S:=(\sum_{i=1}^{N} x_i^2=1)$ coincides with some vector field of the form $V(x)=\sum_{k=1}^{N} P_k(x)V_k(x)$ when restricted to S where $P_1,\dots,P_n: R^n \to R$ are appropriate polynomials.
proof. Let $x=\sum x_i\, e_i$ be fixed, and let $\alpha_x: R^N \to R^N$ be the linear mapping $y \to y-\langle x,y\rangle x$. Consider the operation $\beta_x: y \to (1-\langle x,x\rangle)y^+ + \langle x,y\rangle x$ where y^+ stands for adjoint of y. Observe that

$$\alpha_x(\beta_x(y)) = \beta_x(y) - \langle x,\beta_x\rangle x =$$
$$= (1-\langle x,x\rangle)y + \langle x,y\rangle x - \langle x,(1-2x,x\rangle)y + \langle x,y\rangle x)x =$$
$$= (1-\langle x,x\rangle)y + \langle x,y\rangle x - \langle x,y\rangle x + \langle x,x\rangle\langle x,y\rangle x - \langle x,x\rangle\langle x,y\rangle x = (1-\langle x,x\rangle)y.$$

Then, by writing $V_k(x)=\alpha_x(e_x)$ and $Q^X(x)=\beta_x\big(Q(x)\big)=\sum_{k=1}^{N} q_k^* e_k$ and

$$\alpha_x(Q^*(x)) = \alpha_x(\beta_x(Q(x)) = (1-\langle x,x\rangle)Q(x),$$
$$\alpha_x(\textstyle\sum_{k=1}^{N} q_k^* e_k) = \sum_{k=1}^{N} q_k^* \alpha_x(e_k) = \sum_{k=1}^{N} q_k^*(x)V_k(x).$$

Then, by writing $v_k(x):=\langle V(x),e_k\rangle$ $(k=1,\dots,N)$ for the component function of the vector field V, we have $V(x):=\sum v_k e_k$ and $R(x)-\langle x,R(x)\rangle x = \alpha_x(R(x)) = \alpha_x(\sum_{k=1}^{N} v_k(x)e_k) =$

$$= \sum v_k(x)\alpha(e_k) = \sum_{k=1}^{N} v_k\,(x)V_k(x)$$

By this we get

$$(1-\langle x,x\rangle)Q(x) = \sum q_k^*(x)v_k(x).$$

Thus, with the scalar valued polynomials $p_k(x) \coloneqq v_k(x)+q_k(x)$ we have

$$V(x) = \sum_{k=1}^{N} v_k\,(x)V_k(x) = \sum_{k=1}^{N} q_k^*\,(x)V_k(x) =$$
$$= \sum_{k=1}^{N} (v_k\,(x)+q_k^*(x))V_k(x) =$$
$$= \sum_{k=1}^{N} q_k(x)\,V_k(x) \quad \blacksquare$$

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