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BERNOULLI DIFFERENTIAL EQUATION OF SECOND ORDER WITH FRACTIONAL DERIVATIVE

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الملخص:

في هذا البحث نقدم معادلة برنولي اللاخطية من الدرجة الثانية والتي تشتمل على المشتقات من الرتبة الكسرية. وقد تم استخدام طريقة بيكارد التكرارية والتي تعتبر من أحد الطرق العددية الكلاسكسية لإيجاد الحل التقريبي لهذه المعادلة واختبار سرعة تقارب الحل التقريبي للحل المضبوط.

Abstract

In this study, we present a second order nonlinear equation with nonlinearity of Bernoulli type, which include fractional order derivatives. We consider the numerical solution of the nonlinear equation using the Picard iteration method, the method seeks to examine the convergence of solutions of this type of equations. The resulting solution showed that the convergence could be increased at each iterate level. However, as the number of iterations increases, there is a rapid rate of convergence of the approximate solution to the analytic solution. All Results obtained with the classical Picard method on the equation and were compared with the exact solution.

Keywords: Fractional derivatives, Bernoulli differential equation of second order, Picard iteration method.

1. Introduction

Linear and Nonlinear ordinary differential equations, we can find their exact solutions in Elementary Differential Equations [7]. The exact solution and numerical solutions of this kind of equations play an important role in physical science and in engineering fields; therefore, there have been attempts to develop new techniques for obtaining analytical solutions, which reasonably approximate the exact solutions. In recent years, many research workers have paid attention to find the solutions of linear and nonlinear differential equations by using various methods. Among these are, the Picard iteration method, and the Adomain decomposition method (ADM) which was introduced by G. Adomian [1- 4] to solve linear and nonlinear differential equations, and It is well known that the Adomian decomposition method and its modifications [17,18] are efficient methods to solve linear and nonlinear ODEs.

Differential equations with fractional order derivative have recently proven to be strong tools in the modeling of many physical phenomena and in various fields of science and engineering. In [16] we constructed the approximate solutions of fractional derivatives of order α, and introduced simple comparison between several methods to found the numerical approximate solutions of the fractional derivatives. Therefore, we applied the Picard iteration method to solve nonlinear differential equations of Bernoulli type with fractional derivatives, this study exhibit that the Picard iteration method is very efficient for nonlinear models, and it results give evidence that high accuracy can be achieved.

2. Preliminaries

According to the Riemann–Liouville approach to fractional calculus, the notation of the fractional integral of order α ($\alpha > 0$) is a natural consequence of the well-known formula (usually attributed to Cauchy) that reduces the calculation of the n -fold primitive of a function f to a single integral of the convolution type. We shall start with the definitions. The Cauchy formula reads:

$$I^n f(x) = \frac{1}{(n-1)!}\int_0^x (x-s)^{n-1} f(s)ds, \quad x > 0, n \in \mathbb{N} \tag{2.1}$$

Definition: The Riemann-Liouville fractional order derivative of order α of the left fractional derivative is defining by:

$${}_aD_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-s)^{n-1-\alpha} f(s)ds, \tag{2.2}$$

where G(.) is the classical gamma function, and for a function f given on the interval $[a, b]$, $n-1 \leq \alpha \leq n$, n is positive integer. In particular when $0 \leq \alpha \leq 1$ then:

$${}_aD_x^\alpha f(x) := \frac{1}{\Gamma(1-\alpha)}\left(\frac{d}{dx}\right) \int_a^x (x-s)^{-\alpha} f(s)ds, \tag{2.3}$$

The corresponding right Riemann-Liouville fractional derivative definition is

$${}_xD_b^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)}\left(\frac{-d}{dx}\right)^n \int_x^b (x-s)^{n-1+\alpha} f(s)ds, \tag{2.4}$$

where: $n-1 \leq \alpha < n$. The derivative of a constant is obtained as non-zero using the above definitions (2.2)-(2.4) which contradict the classical derivative of the constant, which is zero. In 1967 Prof. M.Caputo proposed a modification of the R-L definition of fractional derivative which can overcome this shortcoming of the R-L definition.

Definition: The Liouville–Caputo fractional-order derivative of f is defined in the following form [8].

$${}_a^CD_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s)ds \text{ ; where: } n-1 \leq \alpha < n.$$

In this definition first differentiate $f(x)$, n-times then integrate $n-\alpha$-times. The disadvantage of this method is that differentiable n-times then the α- order derivative will exist, where $n-1 \leq \alpha < n$.

In [11], the author discussed existence, uniqueness, and structural stability of solutions of nonlinear differential equations of fractional order. The following theorems presented in [11] that are very similar to the corresponding classical theorems known in the case of first-order equations.

Theorem 2.1 (existence). Assume that $\boldsymbol{\mathcal{D}} := [a, x_0] \times [y_0^{(0)} - \epsilon, y_0^{(0)} + \epsilon]$ with some $x_0 > 0$ and some, $\varepsilon > 0$ and let the function $f: \boldsymbol{\mathcal{D}} \longrightarrow \mathbb{R}$ be continuous. Furthermore, define $x := min\,\{x_0, (\frac{\varepsilon\Gamma(\alpha+1)}{\|f\|_\infty})^{\frac{1}{\varepsilon}}\}$. Then, there exists a function $y: [a, x] \longrightarrow \mathbb{R}$ solving the initial value problem $D^\alpha(y - T_{m-1}[y])(x) = f(x, y(x))$ with initial conditions:

$y^{(i)}(a) = y_0^{(i)}, i = 0,1, \dots, m-1$, where $T_{m-1}[y]$ is the Taylor polynomial of order $m-1$ for y centered at a.

Theorem 2.2 (uniqueness). Assume that $\boldsymbol{\mathcal{D}} := [a, x_0] \times [y_0^{(0)} - \epsilon, y_0^{(0)} + \epsilon]$ with some $x_0 > 0$ and some $\varepsilon > 0$. Furthermore, let the function $f: \boldsymbol{\mathcal{D}} \longrightarrow \mathbb{R}$ be bounded on $\boldsymbol{\mathcal{D}}$ and fulfill a Lipschitz condition with respect to the second variable; i.e.,

$|f(x,y) - f(x,z)| \leq L|y - z|$ with some constant $L > 0$ independent of x, y and z.

In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. For further readings and details on fractional calculus, we refer to the studies by the authors, [13,14,5,9,10,6.15]

3. Main Results

In this paper, we study fractional differential equations associated to the derivative. We present the following second order nonlinear equation of Bernoulli type with fractional derivative as the form:

$$P(x)D^2y + R(x)D^\alpha y + Q(x)Dy + S(x)y = m\,P(x)\frac{y'^2}{y} + R(x)\frac{y}{\Gamma(1-\alpha)x^\alpha} + f(x)y^m \qquad (3.1a)$$

with initial conditions, $y(a) = y_0, y'(a) = y'_0$ where $P(x) \neq 0, Q(x) \neq 0, m \geq 2$ and also y_0 and y_0' are not equal to zero, where $n \leq \alpha < n+1, n$ is positive integer. We rewrite eq.(3.1a) as the form:

$$P(x)\left(y^{-m}D^2y - m\frac{y'^2}{y}y^{-m}\right) + R(x)\left(y^{-m}D^\alpha y - \frac{y^{1-m}}{\Gamma(1-\alpha)x^\alpha}\right) + Q(x)y^{-m}Dy + S(x)y^{1-m} = f(x) \qquad (3.1b)$$

To find solution for this type of differential equations, we shall reduce the Bernoulli's equation (3.1a) to the linear equation by the transformation $u = y^{1-m}$, and hence (3.1a) will becomes:

$$\frac{1}{1-m}\left(P(x)\frac{d^2u}{dx} + Q(x)\frac{du}{dx} + R(x)D^\alpha u\right) + S(x)u = f(x) \qquad (3.2)$$

subject to the initial conditions:

$$u(a) = y^{1-m}(a), \qquad u'(a) = (1-m)y^{-m}(a)y'(a) \qquad (3.3)$$

For solving (3.2) by the classical Picard iteration, we rewrite this equation as a system of first order equations, as follow:
Let $u_1 = u$ and $u_2 = u'$, then we have:

$$u'_1 = u_2 \;,\; u_1(a) = u_{1,0}$$
$$u'_2 = \frac{1-m}{P(x)}(f(x) - S(x)u_1) - \frac{1}{P(x)}(Q(x)u_2 + R(x)D^{\alpha}u_1) \;, \qquad u_2(a) = u_{2,0} \tag{3.4}$$

where $u_{1,0}$, $u_{2,0}$ are the initial values conditions for the system (3.4), and the equivalent integral equations of this system as follows:

$$u_1 = u_{1,0} + \int_a^x u_2(\tau)d\tau$$
$$u_2 = u_{2,0} + \frac{1-m}{P(x)}\int_a^x (f(\tau) - S(\tau)u_1)d\tau - \frac{1}{P(x)}\int_a^x (Q(\tau)u_2 + R(\tau)D^{\alpha}u_1)d\tau \tag{3.5}$$

In view of the Picard iteration method, we construct the following iteration formulation:

$$u_{1,n} = u_{1,0} + \int_a^x u_{2,n-1}(\tau)d\tau \;\;,\;\; n = 1,2,\dots \tag{3.6}$$

$$u_{2,n} = u_{2,0} + \frac{1-m}{P(x)}\int_a^x \big(f(\tau) - S(\tau)u_{1,n-1}\big)d\tau - \frac{1}{P(x)}\int_a^x \big(Q(\tau)u_{2,n-1} + R(\tau)D^{\alpha}u_{1,n-1}\big)d\tau$$

In this paper, one numerical method presented for solving nonlinear Bernuolli equation with fractional derivative; where the approximate solution for eq. (3.1) is obtained from:

$$y = \sqrt[(1-m)]{u} = \sqrt[(1-m)]{u_{1,i}} \;,\; i = 0,1,\dots.$$

In particular, we will be obtaining Bernoulli equation of second order when $Q(x) = 0$ in eq.(3.1). Consequently, we will be repeating this method to find numerical solution for this type of the nonlinear Bernoulli equations and the results are perfect. (This my study unpublished yet). So, we will be previewing the solutions for the eq.(31) in the following examples.

4. EXAMPLES

Example 1. Consider the following second order of nonlinear Bernuolli equation with fractional derivative:

$$y'' - y' - D^{\frac{1}{2}}y = 2\frac{y'^2}{y} - \frac{y}{\sqrt{\pi\, x}} + (\frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}} + \frac{1}{\sqrt{\pi\, x}} + 2x - 2)y^2 \tag{4.1}$$

with initial conditions: $y(0) = 1, y'(0) = 0$, and $\alpha = \frac{1}{2}, m = 2$

To find solution for nonlinear differential equation, we shall reduce the Bernoulli's equation to the linear equation by the transformation $u = y^{-1}$, hence the equation will become:

$$\frac{1}{(-1)}\left(\frac{d^2u}{dx} - \frac{du}{dx} - D^{\frac{1}{2}}u\right) = -(2 - \frac{1}{\sqrt{\pi\, x}} - 2x - \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}})$$
$$\frac{d^2u}{dx} - \frac{du}{dx} - D^{\frac{1}{2}}u = 2 - \frac{1}{\sqrt{\pi\, x}} - 2x - \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}} \tag{4.2}$$

subject to the initial conditions: $u(0) = y^{-1}(0) = 1, u'(0) = (-1)y^{-2}(0) = 0$

Consequently, we rewrite this equation as a system of first order differential equation, as the form:

$$u'_1 = u_2 \ , \ u_{1,0} = 1$$

$$u'_2 = 2 - \frac{1}{\sqrt{\pi\, x}} - 2x - \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}} + u_2 + D^{\frac{1}{2}} u_1 \ , \qquad u_{2,0} = 0$$

and the equivalent integral equations of this system as follows:

$$u_1 = u_{1,0} + \int_0^x u_2(\tau) d\tau$$

$$u_2 = u_{2,0} + \int_0^x \left(2 - \frac{1}{\sqrt{\pi\, \tau}} - 2\tau - \frac{8}{3\sqrt{\pi}} \tau^{\frac{3}{2}}\right) d\tau + \int_0^x \left(u_2(\tau) + D^{\frac{1}{2}} u_1(\tau)\right) d\tau \tag{4.3}$$

Hence, the classical Picard iteration will be taking the formula:

$$u_{1,n} = u_{1,0} + \int_0^x u_{2,n-1}(\tau) d\tau$$

$$u_{2,n} = u_{2,0} + \int_0^x \left(2 - \frac{1}{\sqrt{\pi\, \tau}} - 2\tau - \frac{8}{3\sqrt{\pi}} \tau^{\frac{3}{2}}\right) d\tau + \int_0^x \left(u_{2,n-1}(\tau) + D^{\frac{1}{2}} u_{1,n-1}(\tau)\right) d\tau \tag{4.4}$$

The results in the following tables are showing: in table (1): the first reiterations for solution of linear differential equation with fractional derivative (4.2) which are obtained by using Picard's method, in table (2): shows the results of the approximate solution for second order of nonlinear Bernuolli equation with fractional derivative (4.1), and comparing the eleventh iteration which obtained by Picard method with the exact solution. The results showed that the Picard iteration method is remarkably effective and performing is very easy.

Iteration number	$u_{1,n}$
0	0
1	1
2	$1 + x^2 - \frac{x^3}{3} - \frac{32}{105\sqrt{\pi}} x^{\frac{7}{2}}$
3	$1 + x^2 - \frac{32}{105\sqrt{\pi}} x^{\frac{7}{2}} - \frac{x^4}{12} - \frac{64 x^{\frac{9}{2}}}{945\sqrt{\pi}}$
4	$1 + x^2 - \frac{32 x^{\frac{7}{2}}}{105\sqrt{\pi}} - \frac{x^4}{12} - \frac{64 x^{\frac{9}{2}}}{945\sqrt{\pi}}$

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Table 1

x_i	Exact u	PI: $u_{1,11}$	$\|u - u_{PI}\|$	Exact y	PI: $y_{1,11}$	$\|y - y_{PI}\|$
0.0		1.0	0.00			0.00
0.1		1.01	1.77636×10^{-15}	9	0.9900990099	1.77636×10^{-15}
0.2		1.04	1.61204×10^{-12}	8	0.9615384615	1.49047×10^{-12}
0.3		1.09	9.4293×10^{-11}	1	0.9174311927	7.93645×10^{-11}
0.4		1.16	1.71847×10^{-9}	9	0.8620689668	1.27711×10^{-9}
0.5		1.25	1.64883×10^{-8}		0.8000000106	1.05525×10^{-8}
0.6		1.36	1.05295×10^{-7}	4	0.7352941746	5.69283×10^{-8}
0.7		1.49	5.07342×10^{-7}	1	0.6711411681	2.28522×10^{-7}
0.8		1.64	1.98814×10^{-6}	6	0.6097568367	7.39196×10^{-7}
0.9		1.80999	6.65127×10^{-6}	6	0.5524882181	2.03025×10^{-6}
1.0		1.99998	0.0000196372		0.5000049094	4.90935×10^{-6}

Table 2: shows the approximate solutions for Eq. (4.1)

The following Figures represent the graphical presentation of u and y, where we compare the Picard iteration method $u_{1,11}$ with the exact solution u for linear differential equation (4.2) on the graph (1), then we compare between the Picard method $y_{1,11}$ with the exact solution for nonlinear Bernoulli equation of second order y (4.1) on the graph(2). Numerical simulation shows that $u_{1,11}$ and $y_{1,11}$ both grow with exact solutions.

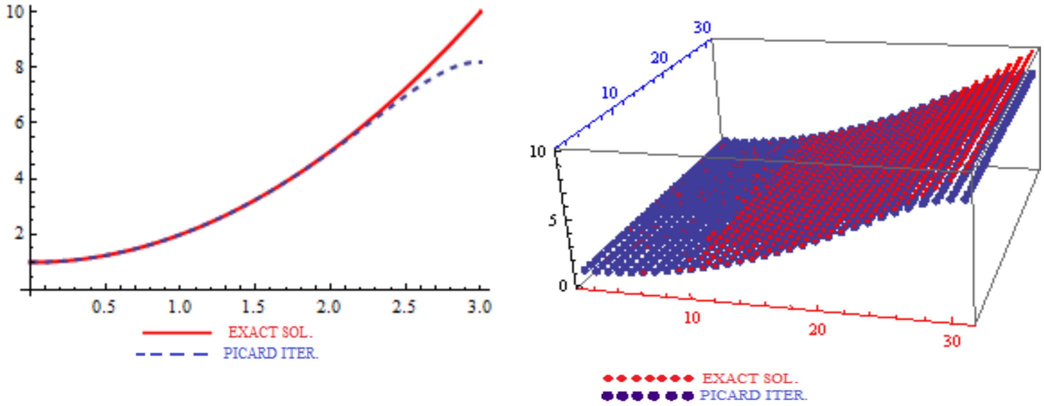


Figure 1: comparing the approximate solution of $u_{1,11}$ with u

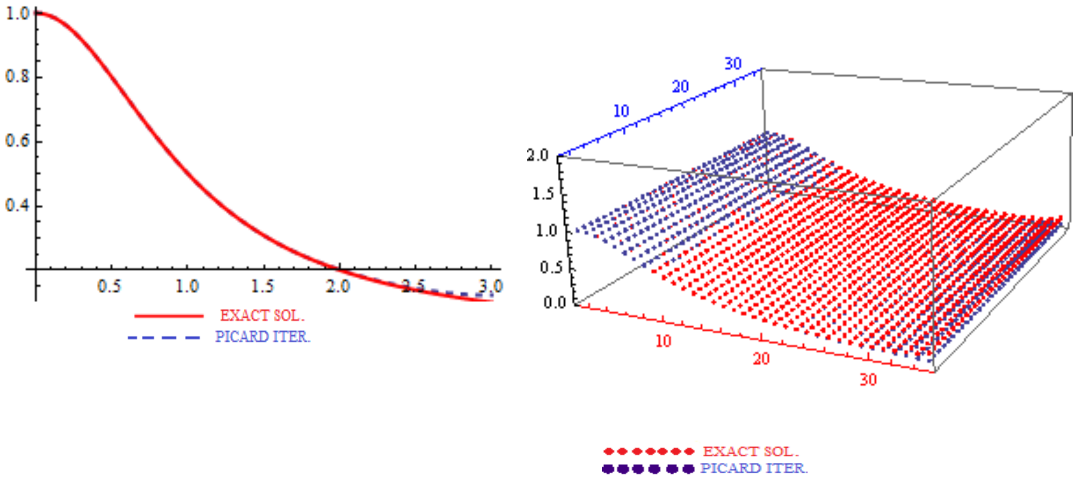


Figure 2: comparing the approximate solution of $y_{1,11}$ with y

Example 2. Consider the following second order of nonlinear Bernuolli equation with fractional derivative:

$$y'' + x\,y' + \frac{\sqrt{\pi}}{2} D^{\frac{1}{2}} y + \frac{1}{2} y = 3\frac{y'^2}{y} + \frac{y}{2\sqrt{x}} + \left(\frac{\sqrt{x}}{2} + \frac{3\,\pi}{16} x - \frac{1}{8\sqrt{x}} - \frac{5}{12} x^{\frac{3}{2}} - \frac{x^2}{2}\right) y^3 \qquad (4.5)$$

with initial conditions: $y(1) = 1, y'(1) = \frac{1}{4}, y''(1) = \frac{-7}{16}$, where $m = 3, \alpha = \frac{1}{2}$

To find solution for nonlinear fractional differential equation, we shall reduce the Bernoulli's equation to the linear equation by transformation $u = y^{-2}$, hence the equation will becomes:

$$\frac{d^2u}{dx} + x\frac{du}{dx} + \frac{\sqrt{\pi}}{2} D^{\frac{1}{2}} u - u = \frac{1}{4\sqrt{x}} - \sqrt{x} - \frac{3\,\pi}{8} x + \frac{5}{6} x^{\frac{3}{2}} + x^2 \qquad (4.6)$$

subject to the initial conditions: $u(1) = 1, u'(1) = \frac{1}{2}, u''(1) = \frac{5}{4}$

Consequently, we rewrite this equation as a system of first order differential equation, as the form:

$$u_1 = u_2 \qquad (4.7)$$

$$u_2 = \frac{1}{4\sqrt{x}} - \sqrt{x} - \frac{3\,\pi}{8} x + \frac{5}{6} x^{\frac{3}{2}} + x^2 + u_1 - x\,u_2 - \frac{\sqrt{\pi}}{2} D^{\frac{1}{2}} u_1$$

The classical Picard iteration will be taking the formula:

$$u_{1,n} = u_{1,0} + \int_1^x u_{2,n-1}(\tau) d\tau \qquad (4.8)$$

$$u_{2,n} = u_{2,0} + \int_1^x \left(\frac{1}{4\sqrt{\tau}} - \sqrt{\tau} - \frac{3\,\pi}{8}\tau + \frac{5}{6}\tau^{\frac{3}{2}} + \tau^2 + u_{1,n-1}(\tau) - \tau\, u_{2,n-1}(\tau) - \frac{\sqrt{\pi}}{2} D^{\frac{1}{2}} u_{1,n-1}(\tau) \right) d\tau$$

The results in the following tables are showing: in table (3): the first reiterations for solution of linear differential equation with fractional derivative (4.6) which obtained by using Picard's method, in table (4): the approximate solution for second order of nonlinear Bernuolli equation with fractional derivative (4.5) obtained by Picard method, where $y_{1,n} = u_{1,n}^{\frac{-1}{2}}, n = 0,1,...,$ and $y = u^{\frac{-1}{2}}$. The results showed that the Picard iteration method is remarkably effective and performing is very easy.

Iteration number	$u_{1,n}$
0	0
1	$\frac{3}{2}-\frac{x}{2}$
2	$\frac{877}{420}-\frac{\pi}{8}-\frac{5x}{4}+\frac{3\pi x}{16}-\frac{x^{\frac{3}{2}}}{3}+\frac{x^2}{2}-\frac{4x^{\frac{5}{2}}}{15\sqrt{\pi}}+\frac{x^3}{12}-\frac{\pi x^3}{16}+\frac{2x^{\frac{7}{2}}}{21}+\frac{x^4}{12}$
3	$\frac{19373}{8316}-\frac{3\pi}{20}-\frac{23621x}{15120}+\frac{15\pi x}{64}-\frac{2x^{\frac{3}{2}}}{3}+\frac{3x^2}{4}-\frac{2x^{\frac{5}{2}}}{15}+\frac{x^3}{8}-\frac{3\pi x^3}{32}+\frac{16x^{\frac{7}{2}}}{105}+\frac{8x^{\frac{9}{2}}}{189}-\frac{x^5}{80}+\frac{3\pi x^5}{320}-\frac{4x^{\frac{11}{2}}}{297}-\frac{x^6}{90}$

Table 3

x_i	*Exact* u	*PI:* $u_{1,5}$	$\lvert u - u_{PI} \rvert$	*Exact* y	*PI:* $y_{1,5}$	$\lvert y - y_{PI} \rvert$
1.0	1.0	1.0	0.0	1.0	1.0	1.0
1.1	0.95631026	0.95622845	0.000081815	1.022587762	1.022631507	0
1.2	0.92546586	0.92519479	0.000271064	1.039488762	1.039641025	0
1.3	0.90777194	0.90726465	0.000507297	1.049570513	1.049863906	0
1.4	0.90349766	0.90275036	0.000747297	1.052050249	1.052485603	0
1.5	0.91288269	0.91194000	0.000942687	1.046628396	1.047169214	0
1.6	0.93614229	0.93512403	0.001018260	1.033544228	1.034106790	0
1.7	0.97347118	0.97261049	0.000860688	1.013534299	1.013982650	0
1.8	1.02504658	1.02471547	0.000331106	0.987707152	0.987866713	0
1.9	1.09103073	1.09171100	0.000680268	0.957373731	0.957075405	0
2.0	1.17157287	1.17370699	0.002134117	0.923879533	0.923039219	0

Table 4: shows the approximate solutions for Eq. (4.5)
and fifth iteration obtained from Picard iteration

In the following graphs, we compare the Picard iteration method $u_{1,5}$ with the exact solution u for linear system (4.8) on the graph (3), then we compare between the Picard method $y_{1,5}$ with the exact solution for nonlinear Bernoulli equation with fractional derivative y on the graph(4).

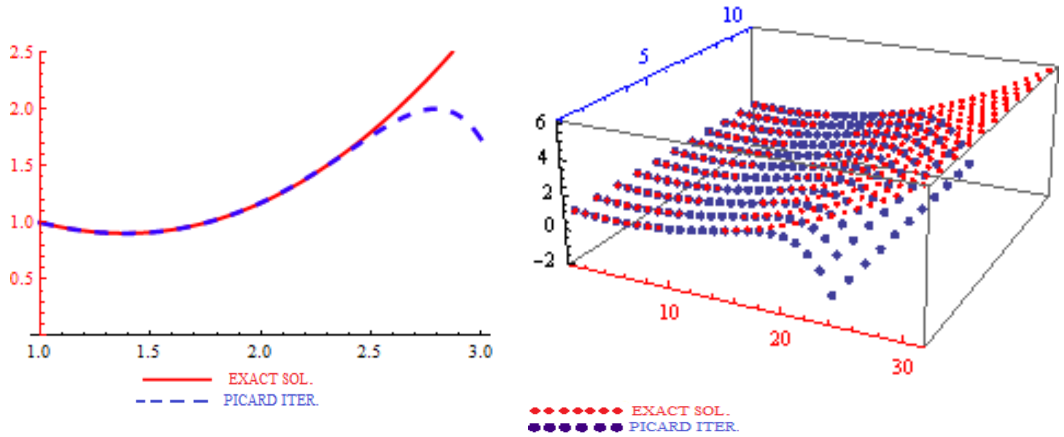
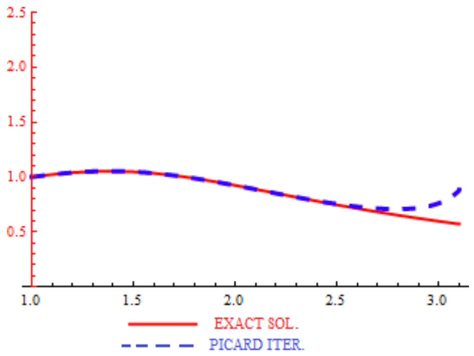


Figure 3: comparing the approximate solution of $u_{1,5}$ with u



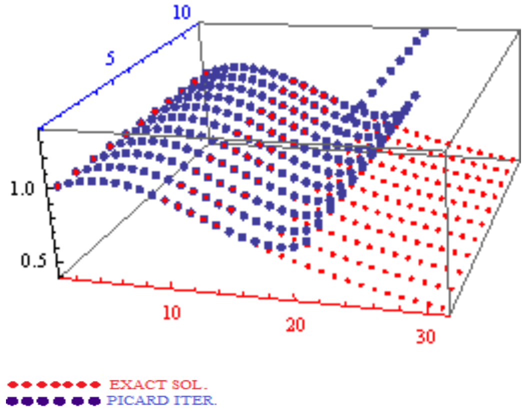


Figure 4: comparing the approximate solution of $y_{1,5}$ with y

Example 3. Consider the following second order of nonlinear Bernuolli equation with fractional derivative:

$$y'' + y' - xD^{\frac{3}{2}}y = 2\frac{y'^2}{y} - \frac{y}{\Gamma(\frac{-1}{2})x^{\frac{3}{2}}} + (\frac{4}{\sqrt{\pi}}x^{\frac{3}{2}} + \frac{8}{\sqrt{\pi}}x^{\frac{5}{2}} - 2 - 8x - 3x^2)y^2 \quad (4.9)$$

with initial conditions: $y(1) = \frac{1}{2}, y'(1) = -\frac{5}{4}, y''(1) = \frac{17}{4}$, where $m = 2$, $\alpha = \frac{3}{2}$, $\Gamma\left(\frac{-1}{2}\right) = -\sqrt{\pi}$.

To find a solution for nonlinear fractional differential equation, we shall reduce the Bernoulli's equation to the linear equation by transformation $u = y^{-1}$, hence the equation will becomes:

$$\frac{d^2u}{dx} + \frac{du}{dx} - xD^{\frac{3}{2}}u = 2 + 8x + 3x^2 - \frac{4}{\sqrt{\pi}}x^{\frac{3}{2}} - \frac{8}{\sqrt{\pi}}x^{\frac{5}{2}} \quad (4.10)$$

subject to the initial conditions: $u(1) = 2$, $u'(1) = 5$, $u''(1) = 8$

Consequently, the classical Picard iteration will be taking the formula:

$$u_{1,n} = u_{1,0} + \int_1^x u_{2,n-1}(\tau)d\tau \quad (4.11)$$

$$u_{2,n} = u_{2,0} + \int_1^x \left(2 + 8\tau + 3\tau^2 - \frac{4}{\sqrt{\pi}}\tau^{\frac{3}{2}} - \frac{8}{\sqrt{\pi}}\tau^{\frac{5}{2}} - u_{2,n-1}(\tau) + x\, D^{\frac{3}{2}}u_{1,n-1}(\tau)\right)d\tau$$

The results in the following tables are showing: in table (5): the first reiterations for solution of linear differential equation with fractional derivative (4.10) which obtained by using Picard's method, in table (6): the approximate solution for second order of nonlinear Bernuolli equation with fractional derivative (4.9) obtained by Picard method $y_{1,n} = u_{1,n}^{-1}$, $n = 0,1,\ldots$, where $y = u^{-1}$. The results showed that the Picard iteration method is remarkably effective and performing is very easy.

Iteration number	$u_{1,n}$
0	0
1	$-3+5x$
2	$-\frac{13}{12}-\frac{226}{63\sqrt{\pi}}+3x+\frac{206x}{35\sqrt{\pi}}-\frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}-\frac{3x^{2}}{2}+\frac{4x^{3}}{3}-\frac{16x^{\frac{7}{2}}}{35\sqrt{\pi}}+\frac{x^{4}}{4}-\frac{32x^{\frac{9}{2}}}{63\sqrt{\pi}}$
3	$-\frac{17}{60}-\frac{4478}{3465\sqrt{\pi}}+\frac{13x}{12}+\frac{359x}{315\sqrt{\pi}}+\frac{2x^{\frac{3}{2}}}{\sqrt{\pi}}-\frac{x^{2}}{2}-\frac{103x^{2}}{35\sqrt{\pi}}+\frac{28x^{\frac{5}{2}}}{15\sqrt{\pi}}+\frac{11x^{3}}{6}-\frac{16x^{\frac{7}{2}}}{35\sqrt{\pi}}-\frac{x^{4}}{12}-\frac{128x^{\frac{9}{2}}}{315\sqrt{\pi}}-\frac{x^{5}}{20}-\frac{64x^{\frac{11}{2}}}{693\sqrt{\pi}}$

Table5

x_i	Exact u	PI: $u_{1,7}$	$\lvert u - u_{PI} \rvert$	Exact y	PI: $y_{1,7}$	$\lvert y - y_{PI} \rvert$
1.0	0.2	2.0	0	0	0	0
1.1	2.541	2.53896894	0	0	0	0
1.2	3.168	3.15856526	0	0	0	0
1.3	3.887	3.86244936	0	0	0	0
1.4	4.704	4.65368227	0	0	0	0
1.5	5.625	5.53460852	0	0	0	0
1.6	6.656	6.50677951	0	0	0	0
1.7	7.803	7.57096136	0	0	0	0
1.8	9.072	8.72728412	0	0	0	0
1.9	10.469	9.97560204	0	0	0	0
2.0	12.0	11.3161489	0	0	0	0

Table 6: shows the approximate solutions for Eq. (4.9)
and the seventh iteration obtained from the Picard iteration

In the following graphs, we compare the Picard iteration method $u_{1,7}$ with the exact solution u for linear system (4.11) on the graph (5), then we compare between the Picard method $y_{1,7}$ with the exact solution for nonlinear Bernoulli equation with fractional derivative y on the graph (6)

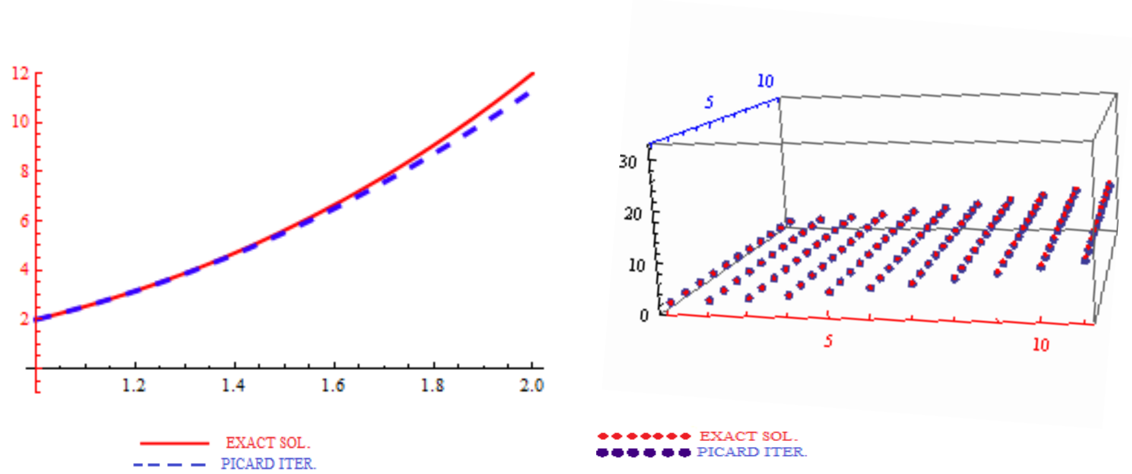


Figure 5: comparing the approximate solution of $u_{1,7}$ with u

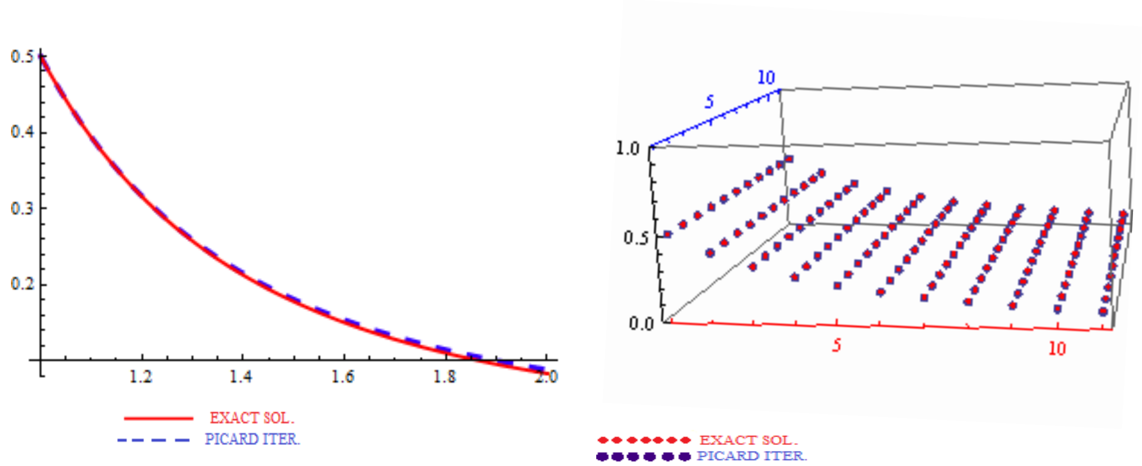


Figure 6: comparing the approximate solution of $y_{1,7}$ with y

Example 4. Consider the following second order of nonlinear Bernuolli equation with fractional derivative:

$$y'' + x^{\frac{2}{3}} D^{\frac{4}{3}} y - y' = 2\frac{y'^2}{y} - \frac{y}{\Gamma\left(\frac{-1}{3}\right) x^{\frac{4}{3}}} + \left(x^2 - 2x - \frac{2}{\Gamma\left(\frac{8}{3}\right)} x^{\frac{7}{3}}\right) y^2 \qquad (4.12)$$

with initial conditions: $y(1) = 3, y'(1) = -9$, where$m = 2, \alpha = \frac{4}{3}$.

To find solution for nonlinear fractional differential equation, we shall reduce the Bernoulli's equation to the linear equation by transformation $u = y^{-1}$, hence the equation will becomes:

$$\frac{d^2u}{dx} + x^{\frac{2}{3}} D^{\frac{4}{3}} u - \frac{du}{dx} = 2x - x^2 + \frac{2}{\Gamma\left(\frac{8}{3}\right)} x^{\frac{7}{3}} \qquad (4.13)$$

subject to the initial conditions : $u(1) = \frac{1}{3},\ u'(1) = 1$

Consequently, we rewrite this equation as a system of first order differential equation, as the form:

$$u_1 = u_2$$
$$u_2 = 2x - x^2 + \frac{2}{\Gamma\left(\frac{8}{3}\right)} x^{\frac{7}{3}} + u_2 - x^{\frac{2}{3}} D^{\frac{4}{3}} u_1 \qquad (4.14)$$

The classical Picard iteration will be taking the formula:

$$u_{1,n} = u_{1,0} + \int_1^x u_{2,n-1}(\tau) d\tau \quad ; n = 1,2, \ldots \qquad (4.15)$$

$$u_{2,n} = u_{2,0} + \int_1^x \left(2\tau - \tau^2 + \frac{2}{\Gamma\left(\frac{8}{3}\right)} \tau^{\frac{7}{3}} + u_{2,n-1}(\tau) - \tau^{\frac{2}{3}} D^{\frac{4}{3}} u_{1,n-1}(\tau)\right) d\tau$$

The results in the following tables are showing: in table (9): the first reiterations for solution of linear differential equation with fractional derivative (4.13) which obtained by using Picard's method, in table (10): the approximate solution for second order of nonlinear Bernoulli equation with fractional derivative. (4.12) obtained by Picard method. The results showed that the Picard iteration method is remarkably effective and performing is very easy.

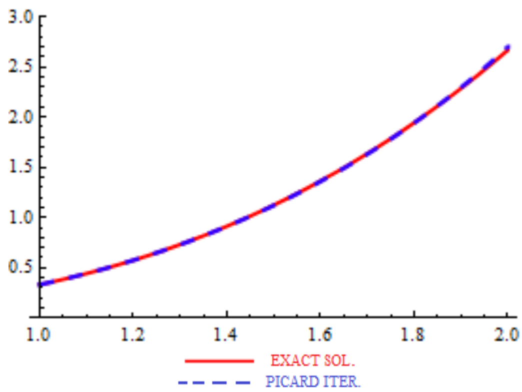
Iteration number	$u_{1,n}$
0	$\frac{1}{3}$
1	$-\frac{2}{3}+x$
2	$\frac{1}{4}-\frac{2x}{3}+\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{12}-\frac{389}{260\Gamma(\frac{-1}{5})}+\frac{131x}{50\Gamma(\frac{-1}{5})}+\frac{3x^{\frac{6}{5}}}{4\Gamma(\frac{-1}{5})}-\frac{243x^{\frac{12}{5}}}{650\Gamma(\frac{-1}{5})}$
3	$-\frac{1}{15}+\frac{x}{4}-\frac{x^2}{3}+\frac{x^3}{2}-\frac{x^5}{60}+\frac{16267}{14560\Gamma(\frac{-1}{5})}-\frac{1341x}{325\Gamma(\frac{-1}{5})}+\frac{3x^{\frac{6}{5}}}{2\Gamma(\frac{-1}{5})}+\frac{131x^2}{100\Gamma(\frac{-1}{5})}+\frac{9x^{\frac{7}{5}}}{14\Gamma(\frac{-1}{5})}-\frac{243x^{\frac{12}{5}}}{650\Gamma(\frac{-1}{5})}-\frac{729x^{\frac{16}{5}}}{10400\Gamma(\frac{-1}{5})}$

Table 7

x_i	*Exact*	$u_{1,8}$	$\lvert u-u_{PI}\rvert$	*Exact* y	*PI:* $y_{1,8}$	$\lvert y-y_{PI}\rvert$
1.0	0.3333333	0.3333333	0.00	3	3	0.00
1.1	0.4436667	0.4436837	0.0000168	2.2539444	2.2538590	0.00008538
1.2	0.576	0.5759277	0.0000723	1.7361111	1.7363291	0.00021794
1.3	0.7323333	0.7318239	0.0005094	1.3654984	1.36644897	0.00095056
1.4	0.9146667	0.9131761	0.0014906	1.0932945	1.09507905	0.00178459
1.5	1.125	1.1219966	0.0030034	0.8888889	0.89126832	0.002379434
1.6	1.3653333	1.3607357	0.0045977	0.7324219	0.73489658	0.002474768
1.7	1.6376666	1.6325896	0.0050769	0.6106249	0.61252377	0.00189887
1.8	1.9440000	1.9418977	0.0021023	0.5144033	0.51496018	0.00055688
1.9	2.2863333	2.2946389	0.0083056	0.4373815	0.43579840	0.00158313
2.0	2.6666667	2.6990436	0.0323769	0.375	0.37050162	0.00449838

Table8: shows the approximate solutions for Eq. (4.12)
and the eight iteration obtained from the Picard method

In the following graphs, we compare the Picard iteration method $u_{1,8}$ with the exact solution u for linear differential equation (4.15) on the graph (7), then we compare between the Picard method $y_{1,8}$ with the exact solution for nonlinear second Bernoulli equation with fractional derivative y on the graph (8).



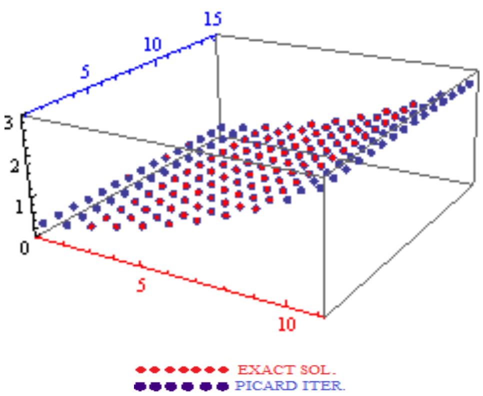


Figure 7: comparing the approximate solution of $u_{1,8}$ with u

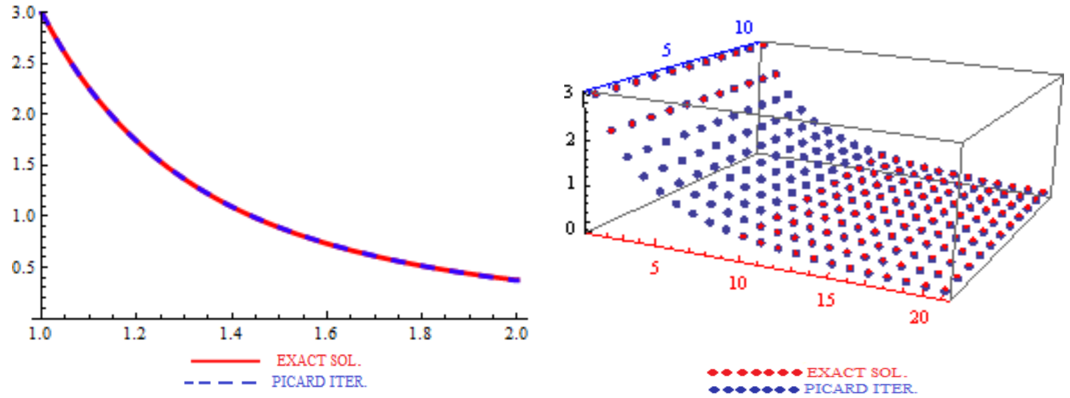


Figure 8: comparing the approximate solution of $y_{1,8}$ with y

5. Conclusion

In this article, we presented a second order nonlinear equation with nonlinearity of Bernoulli type with fractional order derivatives. This study has been extended to the uncompleted previous work. We applied the classical Picard method on the proposed equation to create approximate solutions convergent to the exact solution. Therefore, from the above examples, we conclude that the results obtained by the method are in good agreement with the exact solutions. The study shows that the classical Picard is a reliable technique to solve Bernoulli differential equation involving fractional derivative.

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