Soliton, hyperbolic function, and trigonometric function solutions for $(2+1)$-dimensional coupled Burgers equation

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Abstract

In this work, the modified simple equation method, the $(\frac{G'}{G})$-expansion method, the two variables $(\frac{G'}{G},\frac{1}{G})$-expansion method, and $\tan(\frac{\phi}{2})$-expansion method have been applied to extract new kink soliton, singular soliton, hyperbolic function, and trigonometric function solutions of the $(2+1)$-dimensional coupled Burgers equation. Comparisons of results and the efficiency of the methods have been discussed.

Keywords: $(2+1)$-dimensional coupled Burgers equation, the modified simple equation method, the $(\frac{G'}{G})$-expansion method, the two variables $(\frac{G'}{G},\frac{1}{G})$-expansion method, $\tan(\frac{\phi}{2})$-expansion method, Solitons.

الملخص:

في هذا العمل، تم تطبيق طريقة المعادلة البسيطة المعدلة، طريقة $\left(\frac{G'}{G}\right)$ الممتدة، طريقة المتغيرين $\left(\frac{G'}{G},\frac{1}{G}\right)$ الممتدة و طريقة $\tan\left(\frac{\varphi}{2}\right)$ الممتدة
علي معادلة البرجر المزدوجة ذلت الأبعاد (2+1)
وقد تم استخلاص الحلول التامة و التي ظهرت في صورة

حلول سوليتونية. (kink soliton solution and singular soliton solution)
حلول دوال زائدية و دوال مثلثية (Hyperbolic and trigonometric function solution)
حلول دوال زائدية و دوال مثلثية (Hyperbolic and trigonometric function solution)
حلول دوال كسرية (Rational function solutions)

تم مناقشة مقارنات النتائج و كفاءة الأساليب.

1. Introduction

Solutions of some nonlinear partial differential equations play an important role for understanding many physical phenomena in logical way. One of these equations which arises in various areas such as fluid mechanics, the modeling of gas dynamics, and traffic flow is Burgers equation $[1-5]$

$$u_t = uu_y + avu_x + bu_{yy} + abu_{xx} \tag{1.1}$$

$$u_x = v_y, \tag{1.2}$$

where the subscripts denote differentiations and a and b are constants such that $a \in \mathbb{R}$, $a \neq -1$ and $b \in \mathbb{R}$. In the special case when $a = 2, b = 0.5$ and $u_y = 0$, El-Sabbagh [1] has obtained new various sequences of exact solutions by using combinations of the B$\ddot{a}$cklund transformations and the generalized tanh function expansion method. Kong [2] obtained new explicit exact soliton-like solutions and multi-soliton solutions of equation (1.1) and (1.2) by using the further extended tanh method. Wang [3] constructed a series of exact solutions of equation (1.1) and (1.2) including rational, triangular, periodic wave solutions, rational solitary wave solutions, and rational wave solutions by using a new Riccati equation rational expansion method. Multiple kink solutions and multiple singular kink solutions of equation (1.1) and (1.2) was derived by Wazwaz [4] using Hirota's bilinear method. Yan [5] obtained the variable separation solution with arbitrary number of variable separated function of equation (1.1) and (1.2) by using the multi-linear variable separation approach. Many powerful methods have been applied to extract the exact solutions as well as the soliton solutions for the nonlinear partial differential equations. Some of these methods are the modified simple equation method $[6-10]$, The $(\frac{G'}{G})$-expansion method $[11-14]$, the two variables $(\frac{G'}{G}, \frac{1}{G})$-expansion method $[15-19]$, the improved $\tan(\frac{\phi(\zeta)}{2})$-expansion method $[20-25]$, etc..

The objective of this work is to use four different methods, namely, the modified simple equation method, the $(\frac{G'}{G})$-expansion method, the two variables $(\frac{G'}{G}, \frac{1}{G})$-expansion method, and $\tan(\frac{\phi(\zeta)}{2})$-expansion method to extract new kink soliton, singular soliton, hyperbolic function, and trigonometric function solutions of the $(2+1)$-dimensional coupled Burgers equation. The article is organized as follows: Section 2 describes the four mentioned methods. The exact solution is given in section 3. Applications of these methods are given in section 4. Section 5 is devoted to discussion and conclusion.

2. Description of the modified simple equation method

Assume that we are given nonlinear partial differential equation of the form;

$$W(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0, \tag{2.1}$$

where W is a polynomial function. The main steps for solving equations (1.1) and (1.2) using the modified simple equation method are;

Step 1: We use the wave transformation

$$u = u(\zeta), \quad \zeta = x + y - \mu t, \quad where\ \zeta\ is\ a\ real\ function. \tag{2.2}$$

Step 2: Substituting (2.2) into (2.1) yields an ordinary differential equation in ζ of the form;

$$Q(u, u'(\zeta),, u''(\zeta),, u'''(\zeta), \dots) = 0, \tag{2.3}$$

where Q is a general polynomial.

Step 3: Assume that (2.3) has the formal solution;

$$u(\zeta) = \sum_{i=0}^{N} D_i \left(\frac{\psi'(\zeta)}{\psi(\zeta)}\right)^i, \tag{2.4}$$

where D_i are constants to be determined, such that $D_N \neq 0$, and $\psi(\zeta)$ is an unknown function to be determined later.

Step 4: Determining the positive integer N by balancing the highest order derivatives and the nonlinear terms in equation (2.3).

Step 5: Substituting (2.4) into (2.3) and collecting all the coefficients of $\psi^{-i}(\zeta)$, $i = 0,1,2,3, \dots$ then setting each coefficient to zero, a set of algebraic equations is obtained for $\psi^i(\zeta)$ and D_i. Solving the system we find $\psi^i(\zeta)$, D_i and the exact solution of (2.3).

3. Description of the $\left(\frac{G'}{G}\right)$-expansion method

Assume that we are given a nonlinear partial differential equation of the form;

$$W(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0, \tag{3.1}$$

where W is a polynomial function.

Step 1: We use the wave transformation;

$$u = u(\zeta), \quad \zeta = x + y - \mu t, \quad where\ \zeta\ is\ a\ real\ function, \tag{3.2}$$

to transfer the partial differential equation (3.1) into an ordinary differential equation of the form;

$$Q(u, u'(\zeta), u''(\zeta), u'''(\zeta), \dots) = 0, \tag{3.3}$$

where Q is a general polynomial.

Step 2: Assume that (3.3) has the formal solution;

$$u(\zeta) = \sum_{i=0}^{N} k_i \left(\frac{G'(\zeta)}{G(\zeta)}\right)^i, \tag{3.4}$$

where $G(\zeta)$ satisfies the equation.

$$G'' + \lambda G' + \tau G = 0, \tag{3.5}$$

and k_i, λ, τ are constants to be determined, such that $k_N \neq 0$, and G is the general solution of (3.5) which is of the form

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{\sqrt{\lambda^2-4\tau}}{2}\left(\frac{c_1\cosh(\frac{\sqrt{\lambda^2-4\tau}}{2}\zeta)+c_2\sinh(\frac{\sqrt{\lambda^2-4\tau}}{2}\zeta)}{c_2\cosh(\frac{\sqrt{\lambda^2-4\tau}}{2}\zeta)+c_1\sinh(\frac{\sqrt{\lambda^2-4\tau}}{2}\zeta)}\right)-\frac{\lambda}{2} & ,\lambda^2-4\tau>0, \\ \frac{\sqrt{4\tau-\lambda^2}}{2}\left(\frac{c_1\cos(\frac{\sqrt{4\tau-\lambda^2}}{2}\zeta)-c_2\sin(\frac{\sqrt{4\tau\lambda^2}}{2}\zeta)}{c_2\cos(\frac{\sqrt{4\tau-\lambda^2}}{2}\zeta)+c_1\sin(\frac{\sqrt{4\tau-\lambda^2}}{2}\zeta)}\right)-\frac{\lambda}{2} & ,\lambda^2-4\tau<0. \end{cases} \tag{3.6}$$

Step 3: Determining the positive integer N by balancing the highest order derivatives and the nonlinear terms in equation (3.3).

Step 4: Substitute (3.4) and (3.5) into (3.3) and collect all the coefficients of $(\frac{G'}{G})^i$, $i = 0,1,2,3,\ldots$ then setting each coefficient to zero, a set of algebraic equations are obtained. Solving the system we find λ, τ, μ, and k_i. Substitute back into (3.4) along with (3.6) we get the exact solution of (3.3).

4. Description of the two variable $(\frac{G'}{G},\frac{1}{G})$-expansion method

Consider the equation

$$G''(\zeta)+\lambda G(\zeta)-\tau=0, \tag{4.1}$$

set $\phi=\frac{G'}{G}$, $\psi=\frac{1}{G}$, then we get

$$\phi'=-\phi^2+\tau\psi-\lambda, \qquad \psi'=-\phi\psi. \tag{4.2}$$

The general solution for (4.1) is represented as follows

Case 1: For $\lambda<0$,

$$G(\zeta)=A_1\sinh(\sqrt{-\lambda}\zeta)+A_2\cosh(\sqrt{-\lambda}\zeta)+\frac{\tau}{\lambda},$$

and we have

$$\psi^2=\frac{-\lambda}{\lambda^2(A_1^2-A_2^2)+\tau^2}(\phi^2-2\tau\psi+\lambda). \tag{4.3}$$

Case 2: For $\lambda>0$,

$$G(\zeta)=A_1\sin(\sqrt{\lambda}\zeta)+A_2\cos(\sqrt{\lambda}\zeta)+\frac{\tau}{\lambda},$$

and we have

$$\psi^2=\frac{\lambda}{\lambda^2(A_1^2-A_2^2)-\tau^2}(\phi^2-2\tau\psi+\lambda). \tag{4.4}$$

Case 3: For $\lambda=0$,

$$G(\zeta) = \frac{\tau}{2}\zeta^2 + A_1\zeta + A_2,$$

and we have

$$\psi^2 = \frac{1}{A_1^2 - 2\tau A_2^2}(\phi^2 - 2\tau\psi). \quad (4.5)$$

Assume that we are given nonlinear partial differential equation of the form;

$$W(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0, \quad (4.6)$$

where W is a polynomial function.

Step 1: We use the wave transformation;

$$u = u(\zeta), \quad \zeta = x + y - \mu t, \quad \textit{where } \zeta \textit{ is a real function}, \quad (4.7)$$

to transfer the partial differential equation (4.6) into an ordinary differential equation of the form;

$$Q(u, u'(\zeta), u''(\zeta), u'''(\zeta), \dots) = 0, \quad (4.8)$$

where Q is a general polynomial.

Step 2: Assume that (4.8) has the formal solution;

$$u(\zeta) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \quad (4.9)$$

where a_i, b_i, $i = 1,2,3, \dots, N$ are constants to be determinant.

Step 3: Determining the positive integer N by balancing the highest order derivatives and the nonlinear terms in equation (4.8).

Step 4: Substitute (4.3) for $\lambda < 0$, (4.4) for $\lambda > 0$, (4.5) for $\lambda = 0$ and (4.9) into (4.8) and collect all the coefficients of ϕ and ψ where the degree of ψ is less than or equal to 1. Set each coefficient to zero, a set of algebraic equations are obtained. Solving the system using Matlab we find λ, τ, μ, b_i, and a_i.

5. Description of tan($\frac{\phi}{2}$)-expansion method

Assuming that we are given a nonlinear partial differential equation of the form

$$W(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0, \quad (5.1)$$

where W is a polynomial function. The main steps of $\tan(\frac{\phi}{2})$ expansion method are:

Step 1: Substituting the wave transformation

$$u(x, y, t, \dots) = u(\zeta), \quad \zeta = x + y - \mu t, \quad \textit{where } \zeta \textit{ is a real function}. \quad (5.2)$$

Substituting (5.1) into (5.2) yields an ordinary differential equation in ζ of the form.

$$Q(u, u'(\zeta), u''(\zeta), u'''(\zeta), \dots) = 0, \tag{5.3}$$

where Q is a general polynomial.

Step 2: Assume that (5.3) has the formal solution

$$u(\zeta) = \sum_{i=0}^{N} \delta_i \left[P + tan(\frac{\phi(\zeta)}{2})\right]^i + \sum_{i=1}^{N} \sigma_i \left[P + tan(\frac{\phi(\zeta)}{2})\right]^{-i}, \tag{5.4}$$

where $i = 1,2,3, \dots, N$, δ_i and σ_i are constants to be determined, such that $\delta_N \neq 0$, $\sigma_N \neq 0$ and $\phi = \phi(\zeta)$ satisfies the following equation

$$\phi'(\zeta) = \alpha\sin(\phi(\zeta)) + \beta\cos(\phi(\zeta)) + \gamma$$

Step 3: Determining the positive integer N by balancing the highest order derivatives and the nonlinear term in equation (5.3).

Step 4: Substituting the result into (5.3) and collecting all the coefficients of $\tan(\frac{\phi(\zeta)}{2})^i$ and $\cot(\frac{\phi(\zeta)}{2})^i$ then setting each coefficient to zero, we get a set of algebraic equations.

Step 5: Solve the system using Matlab or Mathematica then substitute the values of δ_0, δ_1,..., δ_N, σ_1, σ_2,..., σ_N, μ, P in (5.4) we get the solution.

6. The exact solution

Using the substitutions $u(x, y, t) = u(\zeta)$ and $v(x, y, t) = v(\zeta)$ where $\zeta = x + y - \mu t$ into equations (1.1) and (1.2) we get

$$-\mu u' = uu' + avu' + bu'' + abu'' \tag{6.1}$$

$$u' = v'. \tag{6.2}$$

Setting $c = 0$ in the integral form $u = v + c$ of equation (6.2) gives

$$\mu u' + (1 + a)uu' + b(1 + a)u'' = 0. \tag{6.3}$$

The exact solution of this equation is thus given by

$$u(\zeta) = v(\zeta) = \frac{\mu}{(1+a)}\left(\tanh(\frac{\mu\zeta}{2b(1+a)}) - 1\right). \tag{6.4}$$

7. On solving (1.1) and (1.2) using the modified simple equation method

Balancing u' and u^2 in equation (6.3) we get $N + 1 = 2N$, i.e. $N = 1$. Substituting into equation (2.4) gives

$$u(\zeta) = D_0 + D_1\left(\frac{\psi'(\zeta)}{\psi(\zeta)}\right), \tag{7.1}$$

which on substituting into (6.3) and collecting all the coefficients of $\psi^{0}(\zeta)$, $\psi^{-1}(\zeta)$ and $\psi^{-2}(\zeta)$ and setting them equal to zero we get a set of algebraic equations in the unknowns D_0, and D_1. Solving this system using Matlab we get

Case 1: For $D_0 = 0$ and $D_1 = 2b$ we have the exact solution

$$u(\zeta) = v(\zeta) = \frac{2\, b\, w_1 e^{-\frac{\mu}{b(1+a)}\zeta}}{w_2 - \frac{w_1 b(1+a)}{\mu} e^{-\frac{\mu}{b(1+a)}\zeta}}, \tag{7.2}$$

where $w_1 = e^{c}$, c is the first integration constant and w_2 is the second integration constant. When $\mu = -b(1+a)$ we have

$$u(\zeta) = v(\zeta) = 2\, b\, w_1 \left[\frac{e^{\zeta}}{w_2 + w_1 e^{\zeta}}\right], \tag{7.3}$$

− If $w_1 = 1$ and $w_2 = 1$, we obtain the kink soliton solution

$$u(\zeta) = v(\zeta) = b\left[1 + \tanh(\tfrac{\zeta}{2})\right], \tag{7.4}$$

− If $w_1 = 1$ and $w_2 = -1$, we obtain the singular soliton solution

$$u(\zeta) = v(\zeta) = b\left[1 + \coth(\tfrac{\zeta}{2})\right], \tag{7.5}$$

Case 2: For $D_0 = \frac{-2\mu}{(1+a)}$ and $D_1 = 2b$ we have the exact solution

$$u(\zeta) = v(\zeta) = -\frac{2\mu}{(1+a)} + \left[\frac{2\, b\, w_1 e^{\frac{\mu}{b(1+a)}\zeta}}{w_2 + \frac{w_1 b(1+a)}{\mu} e^{\frac{\mu}{b(1+a)}\zeta}}\right], \tag{7.6}$$

when $\mu = -b(1+a)$ we get

$$u(\zeta) = v(\zeta) = 2\, b + 2\, b\, w_1 \left[\frac{e^{-\zeta}}{w_2 - w_1 e^{-\zeta}}\right], \tag{7.7}$$

− If $w_1 = -1$ and $w_2 = 1$, we obtain the kink soliton solution

$$u(\zeta) = v(\zeta) = 2\, b - b\left[1 - \tanh(\tfrac{\zeta}{2})\right], \tag{7.8}$$

− If $w_1 = -1$ and $w_2 = -1$, we obtain the singular soliton solution

$$u(\zeta) = v(\zeta) = 2\, b - b\left[1 - \coth(\tfrac{\zeta}{2})\right]. \tag{7.9}$$

8. On solving (1.1) and (1.2) using $(\frac{G'}{G})$-expansion method

Substituting $N = 1$ into equation (3.4) we get

$$u(\zeta) = k_0 + k_1 \left(\frac{G'(\zeta)}{G(\zeta)}\right), \tag{8.1}$$

Substituting into equation (6.3) and collecting all the coefficients of $\left(\frac{G'(\zeta)}{G}\right)^i$, $i = 0,1,2$ setting them equal to zero we obtain a set of algebraic equations in the unknowns λ, τ, k_0, and k_1. Solving this system using Matlab we get

$$\tau = -\tau, \lambda = \pm \frac{\sqrt{4\,a^2\,b^2\tau + 8\,a\,b^2\tau + 4\,b^2\tau + \mu^2}}{(1+a)}, \ k_0 = \frac{b\lambda - \mu}{(1+a)}, \text{ and } k_1 = 2\,b.$$

− For $\lambda^2 - 4\tau > 0$, we get the hyperbolic function solution

$$u(\zeta) = v(\zeta) = \frac{b\lambda - \mu}{(1+a)} + b\sqrt{\lambda^2 - 4\tau}\left(\frac{c_1 \cosh(\frac{\sqrt{\lambda^2-4\tau}}{2}\zeta) + c_2 \sinh(\frac{\sqrt{\lambda^2-4\tau}}{2}\zeta)}{c_2 \cosh(\frac{\sqrt{\lambda^2-4\tau}}{2}\zeta) + c_1 \sinh(\frac{\sqrt{\lambda^2-4\tau}}{2}\zeta)}\right) - \frac{\lambda}{2}. \tag{8.2}$$

− For $\lambda^2 - 4\tau < 0$, we get the trigonometric function solution

$$u(\zeta) = v(\zeta) = -\frac{b\lambda + \mu}{(1+a)} + b\sqrt{4\tau - \lambda^2}\left(\frac{c_1 \cos(\frac{\sqrt{4\tau-\lambda^2}}{2}\zeta) - c_2 \sin(\frac{\sqrt{4\tau\lambda^2}}{2}\zeta)}{c_2 \cos(\frac{\sqrt{4\tau-\lambda^2}}{2}\zeta) + c_1 \sin(\frac{\sqrt{4\tau-\lambda^2}}{2}\zeta)}\right) - \frac{\lambda}{2}. \tag{8.3}$$

9. On solving (1.1) and (1.2) using the two variables $(\frac{G'}{G}, \frac{1}{G})$-expansion method

Substituting $N = 1$ in equation (4.9) we get

$$u(\zeta) = a_0 + a_1\phi + b_1\psi, \tag{9.1}$$

Substituting into equation (6.3) and collecting all the coefficients of ϕ^i ,ψ, and $\phi\psi$ where $i = 0,1,2$ setting them equal to zero yields a set of algebraic equations in the unknowns λ, a_0, a_1, and b_1. Solving this system using Matlab we get

− For $\lambda < 0$, we have the hyperbolic function solutions

Case 1: $\lambda = -\frac{\mu^2}{a^2b^2+2ab^2+b^2}$, $a_0 = -\frac{\mu}{(1+a)}$, $a_1 = b$, $b_1 = \frac{\mu\sqrt{A_1^2 - A_2^2 + 1}}{(1+a)}$,

$$u(\zeta) = v(\zeta) = -\frac{\mu}{(1+a)} + b\left(\frac{G'(\zeta)}{G(\zeta)}\right) + \frac{\mu\sqrt{A_1^2 - A_2^2 + 1}}{(1+a)}\left(\frac{1}{G(\zeta)}\right). \tag{9.2}$$

Case 2: $\lambda = -\frac{\mu^2}{a^2b^2+2ab^2+b^2}$, $a_0 = -\frac{\mu}{(1+a)}$, $a_1 = b$, $b_1 = -\frac{\mu\sqrt{A_1^2 - A_2^2 + 1}}{(1+a)}$,

$$u(\zeta) = v(\zeta) = -\frac{\mu}{(1+a)} + b\left(\frac{G'(\zeta)}{G(\zeta)}\right) - \frac{\mu\sqrt{A_1^2 - A_2^2 + 1}}{(1+a)}\left(\frac{1}{G(\zeta)}\right), \tag{9.3}$$

where $G(\zeta) = A_1 \sinh(\sqrt{-\lambda}\zeta) + A_2 \cosh(\sqrt{-\lambda}\zeta) + \frac{\tau}{\lambda}$, $\tau = -\tau$, and A_1, A_2 are arbitrary constants.

$-$ For $\lambda > 0$, we have the trigonometric function solutions

Case 1: $\lambda = \frac{\mu^2}{a^2b^2+2ab^2+b^2}$, $a_0 = -\frac{\mu}{(1+a)}$, $a_1 = b$, $b_1 = \frac{\mu\sqrt{A_2^2 - A_1^2 + 1}}{(1+a)}$,

$$u(\zeta) = v(\zeta) = -\frac{\mu}{(1+a)} + b\left(\frac{G'(\zeta)}{G(\zeta)}\right) + \frac{\mu\sqrt{A_2^2 - A_1^2 + 1}}{(1+a)}\left(\frac{1}{G(\zeta)}\right). \tag{9.4}$$

Case 2: $\lambda = \frac{\mu^2}{a^2b^2+2ab^2+b^2}$, $a_0 = -\frac{\mu}{(1+a)}$, $a_1 = b$, $b_1 = -\frac{\mu\sqrt{A_2^2 - A_1^2 + 1}}{(1+a)}$,

$$u(\zeta) = v(\zeta) = -\frac{\mu}{(1+a)} + b\left(\frac{G'(\zeta)}{G(\zeta)}\right) - \frac{\mu\sqrt{A_2^2 - A_1^2 + 1}}{(1+a)}\left(\frac{1}{G(\zeta)}\right). \tag{9.5}$$

Where $G(\zeta) = A_1 \sin(\sqrt{\lambda}\zeta) + A_2 \cos(\sqrt{\lambda}\zeta) + \frac{\tau}{\lambda}$, $\tau = -\tau$, and A_1, A_2 are arbitrary constants.

$-$ For $\lambda = 0$, we have $a_0 = -\frac{\mu}{(1+a)}$, $a_1 = 0$, and $b_1 = 0$. In this case we obtained the rejected trivial solution.

10. On solving (1.1) and (1.2) using $\tan(\frac{\phi}{2})$-expansion method

Substituting $N = 1$ in equation (5.4) we have

$$u(\zeta) = \delta_0 + \delta_1[p + \tan(\tfrac{\phi(\zeta)}{2})] + \sigma_1[p + \tan(\tfrac{\phi(\zeta)}{2})]^{-1}. \tag{10.1}$$

Substituting into equation (6.3) and collect all the coefficient of $\tan^n(\frac{\phi(\zeta)}{2})$ setting them equal to zero we obtain a set of algebraic equations in the unknowns δ_0, δ_1, σ_1, p, and μ. Solving this system using Matlab we get

$\delta_0 = b\left[(\alpha + p(\beta - \gamma)) \mp b\sqrt{\alpha^2 + \beta^2 - \gamma^2}\right]$, $\delta_1 = 0$,

$\sigma_1 = b[\beta + \gamma - 2\alpha p - p^2(\beta - \gamma)]$, $p = -p$,

and $\mu = \pm b(1 + a)\sqrt{\alpha^2 + \beta^2 - \gamma^2}$.

Substituting into (10.1) we get

$$u(\zeta) = v(\zeta) = \delta_0 + \frac{\sigma_1}{p+\tan(\frac{\phi(\zeta)}{2})}. \qquad (10.2)$$

Using family 1, 2, 3, 4 and 5, which can be found in [20,21,25], we obtained the following solutions

$$u_1(\zeta) = v(\zeta) = \delta_0 + \frac{\sigma_1}{\left(p+\frac{\alpha}{\beta-\gamma}-\frac{\sqrt{\gamma^2-\beta^2-\alpha^2}}{\beta-\gamma}\tan(\frac{\sqrt{\gamma^2-\beta^2-\alpha^2}}{2}\zeta)\right)}. \qquad (10.3)$$

$$u_2(\zeta) = v(\zeta) = \delta_0 + \frac{\sigma_1}{\left(p+\frac{\alpha}{\beta-\gamma}+\frac{\sqrt{\alpha^2+\beta^2-\gamma^2}}{\beta-\gamma}\tanh(\frac{\sqrt{\alpha^2+\beta^2-\gamma^2}}{2}\zeta)\right)}. \qquad (10.4)$$

$$u_3(\zeta) = v(\zeta) = \delta_0 + \frac{\sigma_1}{\left(p+\frac{\alpha}{\beta}+\frac{\sqrt{\alpha^2+\beta^2}}{\beta}\tanh(\frac{\sqrt{\alpha^2+\beta^2}}{2}\zeta)\right)}. \qquad (10.5)$$

$$u_4(\zeta) = v(\zeta) = \delta_0 + \frac{\sigma_1}{\left(p-\frac{\alpha}{\gamma}+\frac{\sqrt{\gamma^2-\alpha^2}}{\gamma}\tan(\frac{\sqrt{\gamma^2-\alpha^2}}{2}\zeta)\right)}. \qquad (10.6)$$

$$u_5(\zeta) = v(\zeta) = \delta_0 + \frac{\sigma_1}{\left(p+\sqrt{\frac{\beta+\gamma}{\beta-\gamma}}\tanh(\frac{\sqrt{\beta^2-\gamma^2}}{2}\zeta)\right)}. \qquad (10.7)$$

11. Discussion and conclusion

Solutions using the four mentioned methods have been plotted verses the exact solution (6.4) in some selected cases to depict the agreement of results when $\alpha = 1,\ b = 0.1,\ \mu = -0.2,\ y = 1,\ t = 1,$ and $-15 \leq x \leq 15$. Figures (1) (a) represents the kink soliton solution using the modified simple equation method (7.4), verses the exact solution. Figure (1) (b) compares the hyperbolic function solution (8.2) due to the $(\frac{G'}{G})$-expansion method with the exact solution for $\tau = -0.25,\ \lambda = 0,\ c_1 = 0,$ and $c_2 = 1$. Figure (1) (c) depicts the hyperbolic function solution (9.2) due to the $(\frac{G'}{G}, \frac{1}{G})$-expansion method against the exact solution for $\tau = -1,\ \lambda = -1,\ A_1 = 0,$ and $A_2 = 1$. Figure (1) (d) represents the trigonometric function solution (10.6) due to $\tan(\frac{\phi}{2})$-expansion method for $\alpha = 1.28,\ \beta = 0,\ \gamma = 0.8,$ and $p = -15$. Our solutions are considered new compared with other results in $[1-5]$.

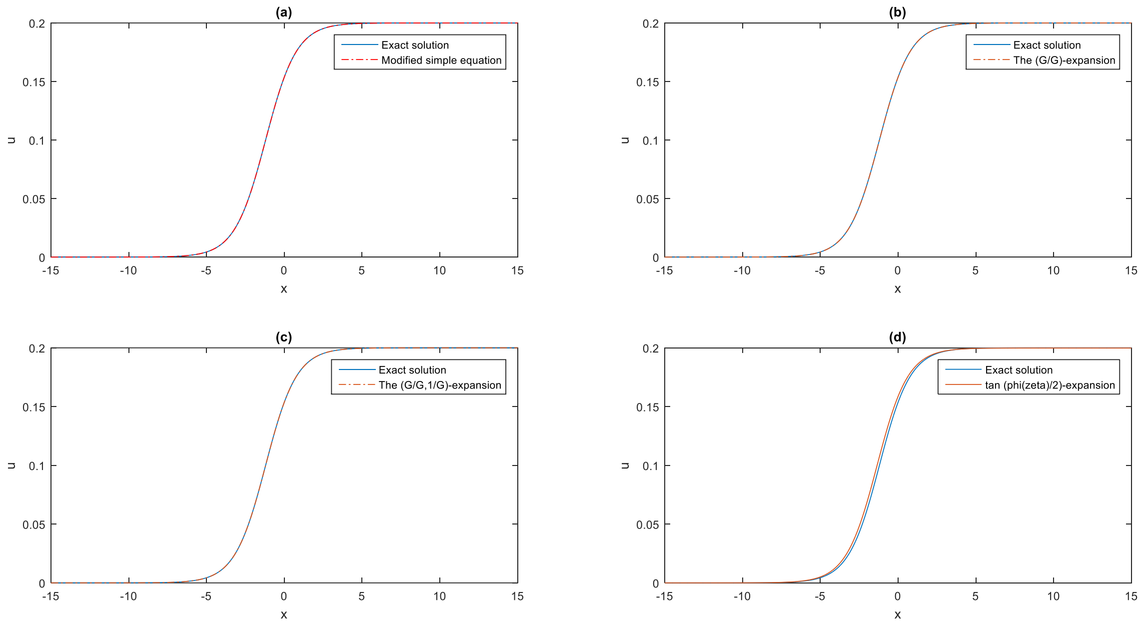

Figure 1: The Exact solution vs: (a) the modified simple equation method, (b) the $(\frac{G'}{G})$-expansion method, (c) the $(\frac{G'}{G}, \frac{1}{G})$-expansion method, (d) the $\tan(\frac{\phi(\zeta)}{2})$.

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