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Analytical Technique Based on New sub-ODE Method for Finding New Optical Solitons and Other Solutions for Lakshmanan-Porsezian-Daniel (LPD) Model

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الملخص:

تم الكشف عن حلول سلوتونية و حلول اخري لنموذج Lakshmanan-Porsezian-Daniel في الالياف ثنائية الانكسار بمساعدة طريقة ODE الفرعية الجديدة. تكشف الطريقة عن حلول سوليتون dark and bright، حلول دورية، حلول كسرية، و حلول Weierstrass and Jacobian elliptic function solutions

Abstract

Solitons and other solutions are revealed for the Lakshmanan-Porsezian-Daniel model in birefringent fibers with the aid of the new sub-ODE method. The prosidure reveal dark and bright soliton solutions, periodic, rational, Weierstrass and Jacobian elliptic function solutions.

Keywords: Birefringent fibers, Lakshmanan-Porsezian-Daniel mode, new sub-ODE, optical solitons.

1. Introduction

The Lakshmanan-Porsezian-Daniel (LPD) model is one of several models which govern the dynamics of soliton transmission across intercontinental distances. This model like other nonlinear partial differential equations has been studied along with strategic algorithms such as trial equation technique, Riccati equation method, modified simple equation method, improved Adomian decomposition method, the method of undetermined coefficients, extended trial function method, $\exp(-\varphi(\zeta))$-expansion method, Jacobi's elliptic function expansion, the $(\frac{G'}{G})$-expansion approach, New mapping method, the modified auxiliary equation method, Extended auxiliary equation approach, and Modified Kudryashov's method $[1-12]$. For more improvements, the new sub-ODE method has been applied to the (LPD) model with birefringence in two component forms. Strategic dark and bright soliton solutions are retrieved. Periodic, rational, Weierstrass and Jacobian elliptic function solutions are also extracted

1.1 Governing model

In the case of birefringent fibers, the (LPD) model divided into vector-coupled equations of the following form:

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + (c_1|q|^2 + d_1|r|^2)q =$$
$$\sigma_1 q_{4x} + (\alpha_1 q_x^2 + \beta_1 r_x^2)q^* + (\gamma_1|q_x|^2 + \delta_1|r_x|^2)q$$
$$+(\lambda_1|q|^2 + \theta_1|r|^2)q_{xx} + (\zeta_1 q^2 + \eta_1 r^2)q_{xx}^*$$
$$+(f_1|q|^4 + g_1|q|^2|r|^2 + h_1|r|^4)q, \tag{1}$$

and

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + (c_2|r|^2 + d_2|q|^2)r =$$
$$\sigma_2 r_{4x} + (\alpha_2 r_x^2 + \beta_2 q_x^2)r^* + (\gamma_2|r_x|^2 + \delta_2|q_x|^2)r$$
$$+(\lambda_2|r|^2 + \theta_2|q|^2)r_{xx} + (\zeta_2 r^2 + \eta_2 q^2)r_{xx}^*$$
$$+(f_2|r|^4 + g_2|r|^2|q|^2 + h_2|q|^4)r, \tag{2}$$

where x and t defines spatial and temporal variables respectively, and the functions $q(x,t)$ and $r(x,t)$ are referring to the wave profiles of the coupled (LPD) model in birefringent fibers. The parameters of a_j and b_j are the group velocity dispersion and spatio-dispersion respectively, while c_j and f_j coefficients correspond to the self-phase modulation, and σ_j is the fourth-order dispersion coefficient where the coefficients of d_j, g_j and h_j are the cross-phase modulation of $j = 1,2$. The remaining terms offer more dispersion impact. To the best of the author knowledge, the system (1) and (2) has not been addressed elsewhere using the mentioned method that we are adressed in this article.

2.Mathematical preliminaries

The initial hypothesis for solving the considered coupled system is

$$q(x,t) = w_1(\zeta(x,t))e^{i\theta(x,t)}, \tag{3}$$

$$r(x,t) = w_2(\zeta(x,t))e^{i\theta(x,t)}, \tag{4}$$

where ζ_j represent the amplitude component of the soliton and θ is the phase component of the soliton that is described as

$$\zeta(x,t) = x - \nu t, \tag{5}$$

$$\theta(x,t) = -kx + \mu t + \zeta_0. \tag{6}$$

Here, ν is the velocity of the soliton, k is the frequency of the solitons in each of the two components while w is the soliton wave number and ζ_0 is the phase constant. Putting (4) and (5) into (1) and (2) we get

$$\begin{aligned}
&(a_1 - b_1\nu + 6k^2\sigma_1)w_1'' + (d_1 + k^2(\beta_1 - \delta_1 + \eta_1 + \theta_1))w_1w_n^2 \\
+\ &(c_1 + k^2(\alpha_1 - \gamma_1 + \lambda_1 + \zeta_0)w_1^3 - (\mu + a_1k^2 + k^2\sigma_1 - b_1k\mu)w_1 \\
-\ &(\beta_1 + \delta_1)w_1(w_n')^2 - f_1w_1^5 - g_1w_1^3w_n^2 - (\alpha_1 + \gamma_1)w_1(w_1')^2 \\
-\ &(\lambda_1 + \zeta_0)w_1^2w_1'' - (\eta_1 + \theta_1)w_n^2w_1'' - h_1w_1w_n^4 - \sigma_1w_1^{(4)} \\
-\ &i[2k(\alpha_1 + \lambda_1 - \zeta_0)w_1^2w_1' + 2k\beta_1w_1w_nw_n' + 4k\sigma_1w_1''' \\
-\ &(\nu + 2a_1k - b_1(k\nu + \mu) + 4k^3\sigma_1)w_1' - 2k(\eta_1 - \theta_1)w_n^2w_1'] = 0
\end{aligned} \tag{7}$$

$$\begin{aligned}
&(a_2 - b_2\nu + 6k^2\sigma_2)w_2'' + (d_2 + k^2(\beta_2 - \delta_2 + \eta_2 + \theta_2))w_2w_n^2 \\
+\ &(c_2 + k^2(\alpha_2 - \gamma_2 + \lambda_2 + \zeta_0)w_2^3 - (\mu + a_2k^2 + k^2\sigma_2 - b_2k\mu)w_2 \\
-\ &(\beta_2 + \delta_2)w_2(w_n')^2 - f_2w_2^5 - g_2w_2^3w_n^2 - (\alpha_2 + \gamma_2)w_2(w_2')^2 \\
-\ &(\lambda_2 + \zeta_0)w_2^2w_2'' - (\eta_2 + \theta_2)w_n^2w_2'' - h_2w_2w_n^4 - \sigma_2w_2^{(4)} \\
-\ &i[2k(\alpha_2 + \lambda_2 - \zeta_0)w_2^2w_2' + 2k\beta_2w_2w_nw_n' + 4k\sigma_2w_2''' \\
-\ &(\nu + 2a_2k - b_2(k\nu + \mu) + 4k^3\sigma_2)w_2' - 2k(\eta_2 - \theta_2)w_n^2w_2'] = 0
\end{aligned} \tag{8}$$

Equation (7) and (8) can be gathered as

$$\begin{aligned}
&(a_j - b_j\nu + 6k^2\sigma_j)w_j'' + (d_j + k^2(\beta_j - \delta_j + \eta_j + \theta_j))w_jw_n^2 \\
+\ &(c_j + k^2(\alpha_j - \gamma_j + \lambda_j + \zeta_0)w_j^3 - (\mu + a_jk^2 + k^2\sigma_j - b_jk\mu)w_j \\
-\ &(\beta_j + \delta_j)w_j(w_n')^2 - f_jw_j^5 - g_jw_j^3w_n^2 - (\alpha_j + \gamma_j)w_j(w_j')^2 \\
-\ &(\lambda_j + \zeta_0)w_j^2w_2'' - (\eta_j + \theta_j)w_n^2w_j'' - h_jw_jw_n^4 - \sigma_jw_j^{(4)} \\
-\ &i[2k(\alpha_j + \lambda_j - \zeta_0)w_j^2w_j' + 2k\beta_jw_jw_nw_n' + 4k\sigma_jw_j''' \\
-\ &(\nu + 2a_jk - b_j(k\nu + \mu) + 4k^3\sigma_j)w_j' - 2k(\eta_j - \theta_j)w_n^2w_j'] = 0
\end{aligned} \tag{9}$$

where $j = 1{,}2$ and $n = 3 - j$, using the balancing principle we get $w_j = w_n$

$$\begin{aligned}
&(a_j - b_j\nu + 6k^2\sigma_j)w_j'' - (\mu + a_jk^2 + k^2\sigma_j - b_jk\mu)w_j \\
+\ &(d_j + c_j + k^2(\beta_j - \delta_j + \eta_j + \theta_j + \alpha_j - \gamma_j + \lambda_j + \zeta_0))w_j^3 \\
-\ &(\beta_j + \delta_j + \alpha_j + \gamma_j)w_j(w_j')^2 - (f_j + h_j + g_j)w_j^5 \\
-\ &(\lambda_j + \zeta_0 + \eta_j + \theta_j)w_j^2w_j'' - \sigma_jw_j^{(4)} - i[(2k(\alpha_j + \lambda_j - \zeta_0 + \beta_j \\
-\ &\eta_j + \theta_j))w_j^2w_j' + 4k\sigma_jw_j''' - (\nu + 2a_jk - b_j(k\nu + \mu) + 4k^3\sigma_j)w_j'] = 0
\end{aligned} \tag{10}$$

break down into real and imaginary parts we get

$$\begin{aligned}&\left(\mu + a_j k^2 + k^2\sigma_j - b_j k\mu\right)w_j - \\ &-\left(d_j + c_j + k^2\left(\beta_j - \delta_j + \eta_j + \theta_j + \alpha_j - \gamma_j + \lambda_j + \zeta_0\right)\right)w_j^3 \\ &+(f_j + h_j + g_j)w_j^5 + (\beta_j + \delta_j + \alpha_j + \gamma_j)w_j(w_j')^2 \\ &-\left(a_j - b_j\nu + 6k^2\sigma_j\right)w_j''\left(\lambda_j + \zeta_0 + \eta_j + \theta_j\right)w_j^2 w_j'' + \sigma_j w_j^{(4)} = 0, \end{aligned} \tag{11}$$

$$\begin{aligned}&(\nu + 2a_j k - b_j(k\nu + \mu) + 4k^3\sigma_j)w_j' - \\ &2k\left(\alpha_j + \lambda_j + \theta_j + \beta_j - \zeta_0 - \eta_j\right)w_j^2 w_j' - 4k\sigma_j w_j''' = 0,\end{aligned} \tag{12}$$

from (12) we have

$$\nu = \frac{b_j\mu - 2ka_j}{1 - kb_j}, \quad kb_j \neq 1 \tag{13}$$

$$\alpha_j + \lambda_j + \theta_j + \beta_j - \zeta_0 - \eta_j = 0 \tag{14}$$

and

$$\sigma_j = 0. \tag{15}$$

So equation (11) reduce to

$$A_{1j}w_j - A_{2j}w_j^3 + A_{3j}w_j^5 + A_{4j}w_j(w_j')^2 - A_{5j}w_j'' + A_{6j}w_j^2 w_j'' = 0. \tag{16}$$

Where

$$\begin{aligned} A_{1j} &= \mu + a_j k^2 - b_j k\mu, \\ A_{2j} &= d_j + c_j + k^2\left(\beta_j + \eta_j + \theta_j + \alpha_j + \lambda_j + \zeta_0 - \delta_j - \gamma_j\right), \\ A_{3j} &= f_j + h_j + g_j, \\ A_{4j} &= \beta_j + \delta_j + \alpha_j + \gamma_j, \\ A_{5j} &= a_j - b_j\nu, \\ A_{6j} &= \lambda_j + \zeta_0 + \eta_j + \theta_j. \end{aligned} \tag{17}$$

2. NEW SUB-ODE METHOD

According to this method [11] we assume that Eq. (16) has the formal solution:

$$w_j = \Omega F^m(\tau), \quad \Omega > 0, \tag{18}$$

where m is a parameter and $F(\tau)$ satisfies the equation:

$$(F'(\tau))^2 = AF^{2-2p}(\tau) + BF^{2-p}(\tau) + CF^2(\tau) + DF^{2+p}(\tau) + EF^{2+2p}(\tau), \ p > 0. \tag{19}$$

Where A, B, C, D and E are constants. We determine m in (18) by using the following homogeneous balance method:

$$D(w_j) = m,\ D(w_j^2) = 2m,\ \ldots, D(w_j') = m + p,\ D(w_j'') = m + 2p, \ldots. \quad (20)$$

It is well known that Eq. (19) has the following cases of solutions:

Case 1. If $A = B = D = 0$, then Eq. (19) has a bright soliton solution:

$$F(\tau) = [\sqrt{-\frac{C}{E}} sech(\sqrt{C}\, p\, \tau)]^{\frac{1}{p}},\ C > 0,\ E < 0, \quad (21)$$

a periodic solution

$$F(\tau) = [\sqrt{-\frac{C}{E}} sec(\sqrt{-C}\, p\, \tau)]^{\frac{1}{p}},\ C < 0,\ E > 0, \quad (22)$$

and rational function solution

$$F(\tau) = [\frac{\epsilon}{\sqrt{E}\, p\, \tau}]^{\frac{1}{p}},\ C = 0,\ \epsilon = \pm 1. \quad (23)$$

Case 2. If $B = D = 0$, $A = \frac{C^2}{4E}$, then Eq. (19) has a dark soliton solution:

$$F(\tau) = [\epsilon \sqrt{-\frac{C}{E}} \tanh(\sqrt{C}\, p\, \tau)]^{\frac{1}{p}},\ C > 0,\ E < 0,\ \epsilon = \pm 1, \quad (24)$$

a periodic solution

$$F(\tau) = [\epsilon \sqrt{-\frac{C}{E}} \tan(\sqrt{-C}\, p\, \tau)]^{\frac{1}{p}},\ C < 0,\ E > 0,\ \epsilon = \pm 1, \quad (25)$$

Case 3. If $B = D = 0$, then Eq. (19) has three Jacobian elliptic function solutions:

$$F(\tau) = [\sqrt{-\frac{Cm^2}{E(2m^2-1)}}\ cn(\sqrt{\frac{C}{2m^2-1}}\ p\, \tau)]^{\frac{1}{p}},\ C > 0,\ A = \frac{C^2 m^2 (m^2-1)}{E(2m^2-1)^2}, \quad (26)$$

$$F(\tau) = [\sqrt{-\frac{C}{E(2-m^2)}}\ dn(\sqrt{\frac{C}{2-m^2}}\ p\, \tau)]^{\frac{1}{p}},\ C > 0,\ A = \frac{C^2(1-m^2)}{E(2-m^2)^2}, \quad (27)$$

and

$$F(\tau) = [\sqrt{-\frac{Cm^2}{E(m^2+1)}}\, sn(\sqrt{-\frac{C}{m^2+1}}\, p\, \tau)]^{\frac{1}{p}},\ C < 0,\ A = \frac{C^2m^2}{E(m^2+1)^2}. \tag{28}$$

Case 4. If $A = B = E = 0$, then Eq. (19) has a bright soliton solution:

$$F(\tau) = [-\frac{C}{D} sech^2(\frac{\sqrt{C}}{2}\, p\, \tau)]^{\frac{1}{p}},\ C > 0,\ D < 0, \tag{29}$$

a periodic solution

$$F(\tau) = [-\frac{C}{D} sec^2(\frac{\sqrt{-C}}{2}\, p\, \tau)]^{\frac{1}{p}},\ C < 0,\ D > 0, \tag{30}$$

and rational function solution

$$F(\tau) = [\frac{4}{Dp^2\tau^2}]^{\frac{1}{p}},\ C = 0,\ D < 0. \tag{31}$$

Case 5. If $C = E = 0, D > 0$ then Eq. (19) has Weierstrass elliptic function solution:

$$F(\tau) = [\wp(\frac{\sqrt{C}}{2}\, p\, \tau, g_2, g_3)]^{\frac{1}{p}}, \tag{32}$$

where $g_2 = -\frac{4B}{D}$ and $g_3 = -\frac{4A}{D}$.

Case 6. If $B = D = 0$, then Eq. (19) has the following Weierstrass elliptic function solutions:

$$(F(\tau) = [\frac{1}{E}\wp(p\tau, g_2, g_3) - \frac{C}{3}]^{\frac{1}{2p}}, \tag{33}$$

where $g_2 = \frac{4C^2-12AE}{3}$ and $g_3 = -\frac{4C(-2C^2+9AE)}{27}$.

$$F(\tau) = [\frac{3A}{3\wp(p\tau, g_2, g_3) - C}]^{\frac{1}{2p}}, \quad (34)$$

where $g_2 = \frac{4C^2 - 12AE}{3}$ and $g_3 = -\frac{4C(-2C^2 + 9AE)}{27}$.

$$F(\tau) = [\frac{\sqrt{12A\wp(p\tau, g_2, g_3) + 2A(2C + \pi)}}{12\wp(p\tau, g_2, g_3) + \pi}]^{\frac{1}{p}}, \quad (35)$$

where $g_2 = -\frac{1}{12}(5C\pi + 4C^2 + 33ACE)$, $g_3 = -\frac{4C}{216}(-21C^2\pi + 63AE\pi - 20C^3 + 27ACE)$

and $\pi = \frac{1}{2}(-5C \pm \sqrt{9C^2 - 36ae})$,

$$F(\tau) = [\frac{6\sqrt{A}\wp(p\tau, g_2, g_3) + C\sqrt{A}}{3\wp'(p\tau, g_2, g_3)}]^{\frac{1}{p}}, \quad (36)$$

where $\wp' = \frac{d\wp}{d\tau} = \pm\sqrt{4\wp^3 - g_2\wp - g_3}$, $g_2 = \frac{C^2}{12}$ and $g_3 = \frac{AEC^3}{6}$,

$$F(\tau) = [\frac{3\sqrt{E^{-1}}\wp'(p\tau, g_2, g_3)}{6\wp(p\tau, g_2, g_3) + C}]^{\frac{1}{p}}, \quad (37)$$

where $g_2 = \frac{C^2}{12} + AE$ and $g_3 = \frac{C(36AE - C^2)}{216}$,

Case 7. If $B = D = 0$, $A = \frac{5C^2}{36E}$, then Eq. (19) has a Weierstrass elliptic function solution:

$$F(\tau) = [\frac{C\sqrt{-\frac{15C}{2E}\wp(p\tau, g_2, g_3)}}{3\wp(p\tau, g_2, g_3) + C}]^{\frac{1}{p}}, \quad (38)$$

where $g_2 = \frac{2C^2}{9}$ and $g_3 = \frac{C^3}{54}$. Here g_2 and g_3 are called invariants of the Wererstrass elliptic function.

Case 8. If $A = B = 0$, then Eq. (19) has three positive solutions:

$$F(\tau) = [\frac{1}{\cosh(\sqrt{C}p\tau) - \frac{D}{2C}}]^{\frac{1}{p}}, C > 0, D < 2C, E = \frac{D^2}{4C} - C, \tag{39}$$

$$F(\tau) = [\frac{1}{2}\sqrt{\frac{C}{E}}[1 + \epsilon\tanh(\frac{1}{2}\sqrt{C}p\tau)]]^{\frac{1}{p}}, C > 0, E > 0, D = -2\sqrt{CE}, \epsilon = \pm 1, \tag{40}$$

and

$$F(\tau) = [\frac{1}{\frac{p^2\tau^2}{4} - E}]^{\frac{1}{p}}, C = 0, D = 1, E < 0. \tag{41}$$

Case 9. If $A = B = 0$, $C > 0$, then Eq. (19) has the hyperbolic function solutions:

$$F(\tau) = [\frac{2C sech^2(\frac{\sqrt{C}}{2}\tau p)}{[\sqrt{D^2 - 4CE} - D] sech^2(\frac{\sqrt{C}}{2}\tau p) - 2\sqrt{D^2 - 4CE}}]^{\frac{1}{p}}, D^2 - 4CE > 0, \tag{42}$$

$$F(\tau) = [\frac{2C csch^2(\frac{\sqrt{C}}{2}\tau p)}{[\sqrt{D^2 - 4CE} - D] csch^2(\frac{\sqrt{C}}{2}\tau p) + 2\sqrt{D^2 - 4CE}}]^{\frac{1}{p}}, D^2 - 4CE > 0, \tag{43}$$

Case 10. If $A = B = 0$, $C < 0$, then Eq. (19) has the periodic function solutions:

$$F(\tau) = [\frac{2C sec^2(\frac{\sqrt{-C}}{2}\tau p)}{[\sqrt{D^2 - 4CE} - D] sec^2(\frac{\sqrt{-C}}{2}\tau p) - 2\sqrt{D^2 - 4CE}}]^{\frac{1}{p}}, D^2 - 4CE > 0, \tag{44}$$

$$F(\tau) = [\frac{2C csc^2(\frac{\sqrt{-C}}{2}\tau p)}{[\sqrt{D^2 - 4CE} - D] csc^2(\frac{\sqrt{-C}}{2}\tau p) - 2\sqrt{D^2 - 4CE}}]^{\frac{1}{p}}, D^2 - 4CE > 0, \tag{45}$$

Balancing $w_j^2 w_j''$ and w_j^5 we get

$$2m + m + 2p = 5m \Longrightarrow m = p. \quad (46)$$

So we now formulate the solution (18) as

$$w_j = \Omega F^p(\tau). \quad (47)$$

Substituting (47) along with (19) into Eq. (16) , collecting all the coefficients of F^{np}, $n = 0, \ldots, 5$ we get

$$\begin{aligned}
F^0(\tau) &: \ \Omega\, A_{5j}\, p^2\, B = 0,\\
F^r(\tau) &: \ A_{1j} + \Omega\, A\, p^2 A_{4j} - C\, p^2\, A_{5j} = 0,\\
F^{2r}(\tau) &: \ 2\Omega\, B\, A_{4j} - 3D\, A_{5j} + \Omega^2\, B\, A_{6j} = 0,\\
F^{3r}(\tau) &: \ -\Omega^2\, A_{2j} + (A_{4j} + \Omega\, A_{6j})\, \Omega\, p^2\, C - 2\, p^2\, E\, A_{5j} = 0,\\
F^{4r}(\tau) &: \ (2A_{4j} + 3\,\Omega\, A_{6j})\, p^2\, D = 0,\\
F^{5r}(\tau) &: \ \Omega^3\, A_{3j} + (A_{4j} + 2\,\Omega\, A_{6j})\, p^2\, E = 0.
\end{aligned}$$

(48) **Case 1.** substituting with $A = B = D = 0$ in (48) then solving the resulting algebraic Eqs. we get the following results:

For: $C = C$ and $E = E$ we have

$$\begin{aligned}
A_{1j} &= C\, p^2\, A_{5j},\ A_{2j} = -\frac{4\, p^2 E^2 A_{5j} - C\, E\, p^2\, A_{4j} + C\, A_{3j}}{2E},\\
A_{6j} &= -\frac{E\, p^2\, A_{4j} + A_{3j}}{2E\, r^2},\ \Omega = 1.
\end{aligned} \quad (49)$$

From (17) and (49) we have the wave number

$$\mu = \frac{C\, p^2 A_{5j} - a_j\, k^2}{1 - b_j k},\ b_j\, k \neq 1. \quad (50)$$

And the frequency of solitons

$$k = \sqrt{\frac{CA_{4j}p^2E - 4A_{5j}p^2E^2 - CA_{3j} - 2E(c_j + d_j)}{2E(\alpha_j - \gamma_j + \lambda_j + \zeta_j + \beta_j + \delta_j + \eta_j + \theta_j)}}. \quad (51)$$

Provided

$(CA_{4j}p^2E - 4A_{5j}p^2E^2 - CA_{3j} - 2E(c_j + d_j))[2E(\alpha_j - \gamma_j + \lambda_j + \zeta_j + \beta_j + \delta_j + \eta_j + \theta_j)] > 0.$ (52)

Substituting (49) along with (21) into Eq. (47) , we obtain the following solutions of the coupled system (1) and (2):

when $C > 0, E < 0$ a bright soliton solutions in the form:

$$q(x,t) = \sqrt{-\frac{C}{E}}\,sech\left(\sqrt{C}p\tau\right)e^{i\left(-\sqrt{\frac{CA_{41}p^2E-4A_{51}p^2E^2-CA_{31}-2E(c_1+d_1)}{2E(\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1)}}x+\frac{C\,p^2A_{51}-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (53)$$

$$r(x,t) = \sqrt{-\frac{C}{E}}\,sech\left(\sqrt{C}p\tau\right)e^{i\left(-\sqrt{\frac{CA_{42}p^2E-4A_{52}p^2E^2-CA_{32}-2E(c_2+d_2)}{2E(\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2)}}x+\frac{C\,p^2A_{52}-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (54)$$

when $C < 0, E > 0$ a periodic solutions in the form:

$$q(x,t) = \sqrt{-\frac{C}{E}}\,sec\left(\sqrt{-C}p\tau\right)e^{i\left(-\sqrt{\frac{CA_{41}p^2E-4A_{51}p^2E^2-CA_{31}-2E(c_1+d_1)}{2E(\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1)}}x+\frac{C\,p^2A_{51}-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (55)$$

$$r(x,t) = \sqrt{-\frac{C}{E}}\,sec\left(\sqrt{-C}p\tau\right)e^{i\left(-\sqrt{\frac{CA_{42}\,p^2E-4A_{52}p^2E^2-CA_{32}-2E(c_2+d_2)}{2E(\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2)}}x+\frac{C\,p^2A_{52}-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (56)$$

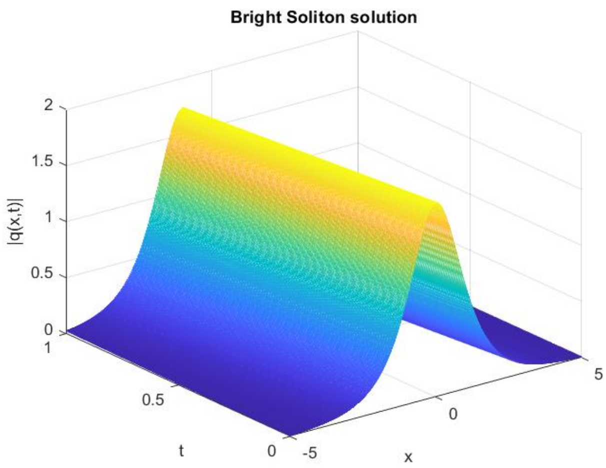

FIGURE 1 THE PLOT OF |Q(X,T)|OF THE SOLUTION (53) WHEN C=1.5, E=-0.5, P=1, B=0.5, M=1, K=1, AND A=0.5

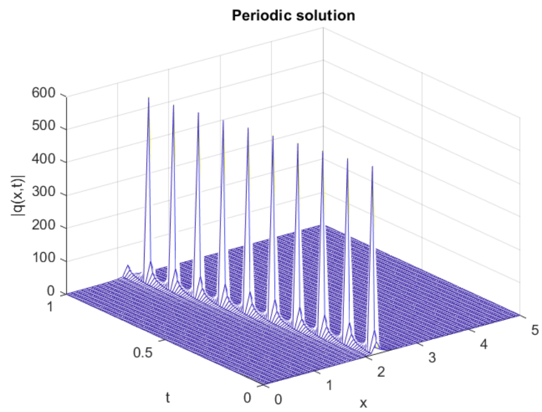

FIGURE 2 THE PLOT OF |Q(X,T)|OF THE SOLUTION (55) WHEN C=-0.5, E=1.5, P=1, B=0.5, M=1, K=1, AND A=0.5

when $C = 0, E > 0$ and $\epsilon = \pm 1$ a rational function solutions in the form:

$$q(x,t) = \frac{\epsilon}{\sqrt{E p\tau}}\, e^{i\left(-\sqrt{\frac{CA_{41}p^2E-4A_{51}p^2E^2-CA_{31}-2E(c_1+d_1)}{2E(\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1)}}x+\frac{C\,p^2A_{51}-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (57)$$

$$r(x,t) = \frac{\epsilon}{\sqrt{E p\tau}}\, e^{i\left(-\sqrt{\frac{CA_{42}p^2E-4A_{52}p^2E^2-CA_{32}-2E(c_2+d_2)}{2E(\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2)}}x+\frac{C\,p^2A_{52}-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}. \quad (58)$$

Case 2. substituting with $B = D = 0$ in (48) then solving the resulting algebraic Eqs. we get the following results:

For: $C = C$ and $E = E$ we have

$$A_{1j} = C\,p^2 A_{5j} - AA_{4j}p^2,\ A_{2j} = CA_{4j}p^2 + CA_{6j}p^2 - 2EA_{5j}p^2,$$
$$A_{3j} = -EA_{4j}p^2 - 2EA_{6j}p^2,\ \Omega = 1. \tag{59}$$

From (17) and (59) we have the wave number

$$\mu = \frac{C\,p^2 A_{5j} - AA_{4j}p^2 - a_j\,k^2}{1 - b_j k},\ b_j\,k \neq 1. \tag{60}$$

And the frequency of solitons

$$k = \sqrt{\frac{CA_{4j}p^2 + CA_{6j}p^2 - 2EA_{5j}p^2 - (c_j + d_j)}{\alpha_j - \gamma_j + \lambda_j + \zeta_j + \beta_j + \delta_j + \eta_j + \theta_j}}. \tag{61}$$

Provided

$$(CA_{4j}p^2 + CA_{6j}p^2 - 2EA_{5j}p^2 - (c_j + d_j))[\alpha_j - \gamma_j + \lambda_j + \zeta_j + \beta_j + \delta_j + \eta_j + \theta_j] > 0. \tag{62}$$

Substituting (59) along with (21) into Eq. (47) , we obtain the following solutions of the coupled system (1) and (2) :

Weierstrass elliptic function solutions

$$q(\mathrm{x,t}) = \left(\frac{1}{E}(p\tau, g_2, g_3) - \frac{C}{3}\right) e^{i\left(-\sqrt{\frac{CA_{41}p^2 + CA_{61}p^2 - 2EA_{51}p^2 - (c_1 + d_1)}{\alpha_1 - \gamma_1 + \lambda_1 + \zeta_1 + \beta_1 + \delta_1 + \eta_1 + \theta_1}}x + \frac{C\,p^2 A_{51} - AA_{41}p^2 - a_1\,k^2}{1 - b_1 k}t + \zeta_0\right)}, \tag{63}$$

$$r(x,t) = \left(\frac{1}{E}(p\tau, g_2, g_3) - \frac{C}{3}\right) e^{i\left(-\sqrt{\frac{CA_{42}p^2 + CA_{62}p^2 - 2EA_{52}p^2 - (c_2 + d_2)}{\alpha_2 - \gamma_2 + \lambda_2 + \zeta_2 + \beta_2 + \delta_2 + \eta_2 + \theta_2}}x + \frac{C\,p^2 A_{52} - AA_{42}p^2 - a_2\,k^2}{1 - b_2 k}t + \zeta_0\right)}. \tag{64}$$

And

$$q(\mathrm{x,t}) = \sqrt{\left(\frac{3A}{3\wp(p\tau,g_2,g_3)-C}\right)}\, e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-AA_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (65)$$

$$r(x,t) = \sqrt{\left(\frac{3A}{3\wp(p\tau,g_2,g_3)-C}\right)}\, e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-AA_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (66)$$

where $g_2 = \frac{4C^2-12AE}{3}$ and $g_2 = \frac{4C(-2C^2+9AE)}{27}$

$$q(x,t) = \left(\frac{\sqrt{12A\wp(p\tau,g_2,g_3)+2A(2C+\pi)}}{12\wp(p\tau,g_2,g_3)+\pi}\right) e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-AA_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (67)$$

$$r(x,t) = \left(\frac{\sqrt{12A\wp(p\tau,g_2,g_3)+2A(2C+\pi)}}{12\wp(p\tau,g_2,g_3)+\pi}\right) e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-AA_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (68)$$

where $g_2 = -\frac{1}{12}(5C\pi + 4C^2 + 33ACE)$, $g_3 = -\frac{4C}{216}(-21C^2\pi + 63AE\pi - 20C^3 + 27ACE)$

and $\pi = \frac{1}{2}(-5C \pm \sqrt{9C^2 - 36AE})$

$$q(x,t) = \left(\frac{6\sqrt{A}\wp(p\tau,g_2,g_3)+C\sqrt{A}}{3\wp'(p\tau,g_2,g_3)}\right) e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-AA_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (69)$$

$$r(x,t) = \left(\frac{6\sqrt{A}\wp(p\tau,g_2,g_3)+C\sqrt{A}}{3\wp'(p\tau,g_2,g_3)}\right) e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-AA_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (70)$$

where $g_2 = \frac{C^2}{12} + AE$, $g_3 = \frac{AEC^3}{6}$ and $\wp' = \sqrt{4\wp^3 - g_2\wp - g_3}$

$$q(x,t) = \left(\frac{3\sqrt{E^{-1}}\wp'(p\tau,g_2,g_3)}{6\wp(p\tau,g_2,g_3)+C}\right) e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-AA_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (71)$$

$$r(x,t) \;\; = \left(\frac{3\sqrt{E^{-1}}\wp'(p\tau,g_2,g_3)}{6\wp(p\tau,g_2,g_3)+C}\right)e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-AA_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (72)$$

where $g_2 = \frac{C^2}{12} + AE$ and $g_3 = \frac{C(36AE-C^2)}{216}$.

When $A = \frac{5C^2}{36E}$ a Weierstrass elliptic function solutions in the form:

$$q(x,t) \;\; = \left(\frac{C\sqrt{-\frac{15C}{2E}}\wp(p\tau,g_2,g_3)}{3\wp(p\tau,g_2,g_3)+C}\right)e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-\frac{5C^2}{36E}A_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (73)$$

$$r(x,t) \;\; = \left(\frac{C\sqrt{-\frac{15C}{2E}}\wp(p\tau,g_2,g_3)}{3\wp(p\tau,g_2,g_3)+C}\right)e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-\frac{5C^2}{36E}A_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (74)$$

where $g_2 = \frac{2C^2}{9}$ and $g_3 = \frac{C^3}{54}$.

When $A = \frac{C^2}{4E}$, $C > 0$, $E < 0$ and $\epsilon = \pm 1$ a dark soliton solutions in the form:

$$q(x,t) \;\; = \left(\epsilon\sqrt{-\frac{C}{E}}\tanh\left(\sqrt{C}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-\frac{C^2}{4E}A_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (75)$$

$$r(x,t) \;\; = \left(\epsilon\sqrt{-\frac{C}{E}}\tanh\left(\sqrt{C}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-\frac{C^2}{4E}A_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (76)$$

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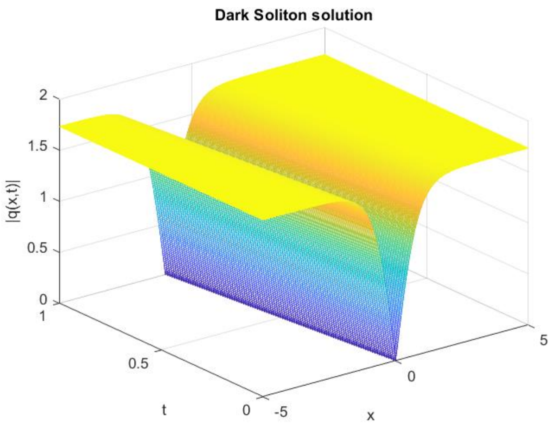

FIGURE 3 THE PLOT OF |Q(X,T)|OF THE SOLUTION (72) WHEN C=1.5, E=-0.5, EPSILON=1 P=1, B=0.5, M=1, K=1, AND A=0.5

when $A=\frac{C^2}{4E}$, $C<0$, $E>0$ and $\epsilon=\pm 1$ a periodic solutions in the form:

$$q(x,t) = \left(\epsilon\sqrt{-\frac{C}{E}}\tan\left(\sqrt{C}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-\frac{C^2}{4E}A_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (77)$$

$$r(x,t) = \left(\epsilon\sqrt{-\frac{C}{E}}\tan\left(\sqrt{C}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-\frac{C^2}{4E}A_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}. \quad (78)$$

Jacobian elliptic function solutions in the form:

$$q(x,t) = \left(\sqrt{-\frac{Cm^2}{E(2m^2-1)}}cn\left(\sqrt{\frac{C}{2m^2-1}}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-AA_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (79)$$

$$r(x,t) = \left(\sqrt{-\frac{Cm^2}{E(2m^2-1)}}cn\left(\sqrt{\frac{C}{2m^2-1}}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-AA_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (80)$$

where $A=\frac{C^2m^2(m^2-1)}{E(2m^2-1)^2}$ and $C>0$

$$q(x,t) = \left(\sqrt{-\frac{Cm^2}{E(2m^2-1)}}\,cn\left(\sqrt{\frac{C}{2m^2-1}}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-AA_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (81)$$

$$r(x,t) = \left(\sqrt{-\frac{Cm^2}{E(2m^2-1)}}\,cn\left(\sqrt{\frac{C}{2m^2-1}}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-AA_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (82)$$

where $A=\frac{C^2(1-m^2)}{E(2-m^2)^2}$ and $C>0$

$$q(x,t) = \left(\sqrt{-\frac{Cm^2}{E(m^2+1)}}\,sn\left(\sqrt{-\frac{C}{m^2+1}}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{41}p^2+CA_{61}p^2-2EA_{51}p^2-(c_1+d_1)}{\alpha_1-\gamma_1+\lambda_1+\zeta_1+\beta_1+\delta_1+\eta_1+\theta_1}}x+\frac{C\,p^2A_{51}-AA_{41}p^2-a_1\,k^2}{1-b_1k}t+\zeta_0\right)}, \quad (83)$$

$$r(x,t) = \left(\sqrt{-\frac{Cm^2}{E(m^2+1)}}\,sn\left(\sqrt{-\frac{C}{m^2+1}}p\tau\right)\right)e^{i\left(-\sqrt{\frac{CA_{42}p^2+CA_{62}p^2-2EA_{52}p^2-(c_2+d_2)}{\alpha_2-\gamma_2+\lambda_2+\zeta_2+\beta_2+\delta_2+\eta_2+\theta_2}}x+\frac{C\,p^2A_{52}-AA_{42}p^2-a_2\,k^2}{1-b_2k}t+\zeta_0\right)}, \quad (84)$$

where $A=\frac{C^2m^2}{E(m^2+1)^2}$ and $C<0$

4. Physical illustrations

In this section, we present numerical simulation of the Lakshmanan-Porsezian-Daniel model (LPD).

The bright soliton solution |q(x,t)| of equation (53) has been depected in Figurs 1 under the selected parameters C=1.5, E=-0.5, p=1, b=0.5, m=1, k=1 and a=0.5. Figrt 2 represents periodic solutions |q(x,t)| of equation (55) under the selected parameters C=-0.5, E=1.5, p=1, b=0.5, m=1, k=1. The representation of Figure 3 indicates dark soliton solution |q(x,t)| of equation (72) under the selected parameters C=1.5, E=-0.5, epsilon=1 p=1, b=0.5, m=1, k=1. These Figures are signifies the dynamic of the selected solutions.

5. Conclusions

The coupled system corresponding to (LPD) equation in birefringent fibers was investigated by using the new sub-ODE method for finding Optical solitons and other solutions. The procedure reveals dark soliton solutions, bright soliton solutions, periodic solutions, rational solutions, Weierstrass elliptic function solutions and Jacobian elliptic function solutions. This method have been applied for the firest time to the coupled system corresponding to (LPD) model in birefringent fibers.

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