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On H-property in bitopological spaces

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الملخص.

في هذا البحث، تم تعريف خاصية H في الفضاءات البتوبولوجية. كما تمت مناقشة خواص الفضاءات مع خاصية Hوالعلاقة بينها وبين الفضاءات $P\text{-}T_0$ و $P\text{-}T_1$ و $P\text{-}T_2$ و P-regular.

Abstract. In this paper, H-property is defined in bitopological spaces. Characterizations of spaces with H-property and the relationship between such spaces and $P\text{-}T_0$, $P\text{-}T_1$, $P\text{-}T_2$ and P-regular spaces are discussed.

Keywords: Axioms separated, H-property, bitopological spaces, P-regular space.

1.Introduction

The concept of bitopological spaces was introduced by Kelly [1] in 1963; where he considered a bitopological space (X, τ_1, τ_2) as a non-empty set X equipped with two topologies τ_1 and τ_2. In that paper, Kelly defined pairwise separation axioms in bitopological spaces as pairwise Hausdorff, pairwise regular and pairwise normal axioms. He also investigated their properties. More details on bitopological spaces can be found in [2-9]. In [10], Murdeshwar and Naimpally introduced the notions of pairwise T_0 and pairwise T_1 spaces. The fivst auther introduced H-property in [11]. In this work, H-property is defined in bitopological spaces characterizations of spaces with H-property and also the relationship between such spaces and $P\text{-}T_0$,

$P\text{-}T_1$, $P\text{-}T_2$ and P-regular spaces are investigated.

2. Bitopological spaces:

2.1. Definition. [10] A bitopological space (X, τ_1 , τ_2) is called $P\text{-}T_0$ space if whenever x and y are distinct points in X there is an open set $U \in \tau_1 \cup \tau_2$ containing one point and not the other.

2.2. Example. Let X be any non empty set. Define τ_1 to be the discrete topology on X and τ_2 is the trivial topology on X, then (X, τ_1 , τ_2) is $P\text{-}T_0$.

2.3. Example. Let $X=\{a,b,c\}$, $\tau_1=\{ X,\phi,\{a,c\}\}$ and $\tau_2 =\{ X,\phi,\{c\}\}$.
Then(X, τ_1 , τ_2) is $P\text{-}T_0$ space.

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2.4. Remark. If (X, τ_1) and (X, τ_2) are T_0 – spaces , then (X, τ_1 , τ_2) is P-T_0 space. In general, the converse is not true. Take X be any infinite set, τ_1 is the trivial topology and τ_2 is the discrete topology .
2.5. Definition. [10] A bitopological space (X, τ_1 , τ_2) is called P-T_1 space if whenever x and y are distinct points in X there are two open set U,V∈ $\tau_1 \cup \tau_2$ one containing x but not y and the other containing y but not x.

2.6. Example. Let X={a,b}, τ_1={ X,ϕ,{a}}and τ_2 ={ X,ϕ,{b}}. Then(X, τ_1 , τ_2) is P-T_1 space.

2.7. Remark. The above example shows that if (X, τ_1 , τ_2) is P-T_1 space then this doesn't imply (X, τ_1) or (X, τ_2) is T_1 – space.

The following theorem shows that P-T_1 ⇒P-T_0.

2.8. Theorem. Let (X, τ_1 , τ_2) be a P-T_1 space. Then it is P-T_0 as well.

The following example shows that P-T_0 space is not necessary P-T_1 space.

2.9. Example. Let X={a,b}, τ_1={ X,ϕ,{a}}and τ_2 ={ X,ϕ}. Then(X, τ_1 , τ_2) is

P-T_0 but is not P-T_1.

2.10. Definition. Let(X, τ_1 , τ_2) be a bitopological space, Y⊂X and let

τ_{1y} ={U∩Y: U∈ τ_1} and τ_{2y} ={V∩Y: V∈ τ_2}.then, τ_{1y} and τ_{2y} are both topologies on Y and (Y, τ_{1y} , τ_{2y}) is called the subspace of the bitopological space (X,τ_1,τ_2).

2.11. Theorem. Let (X, τ_1 , τ_2) be a bitopological space which is P-T_1 (P-T_0) space. Then every subspace of X is P-T_1 (P-T_0) space.

2.12. Definition. A space (X, τ_1 , τ_2) is said to be P-T_2 space if any two distinct points in X can be separated by two disjoint open sets in $\tau_1 \cup \tau_2$.

2.13. Example. Let X be an infinite set, τ_1 is the co-finite topology on X and τ_2 the discrete topology on X . Then, (X, τ_1 , τ_2) is P-T_2 space.

The following theorem states that P-T_2 ⇒P-T_1.

2.14. Theorem. Let (X, τ_1 , τ_2) be a bitopological space which is P-T_2 space, then X is P-T_1 space.

2.15. Theorem. Let (X, τ_1 , τ_2) be a bitopological space which is P-T_2 space. Then, every subspace of X is P-T_2 space.
2.16. Definition. (Kelly) let (X, τ_1 , τ_2) be a bitopological space. Then τ_1 is said to be regular with respect to τ_2 if for each point x in X and each closed set P in τ_1 such that $x \notin$ P there are two sets U ∈ τ_1 and V∈ τ_2 such that x ∈U, P ⊆ V and U∩V = ϕ .

2.17. Definition. A bitopological space (X, τ_1 , τ_2) is said to be P-regular if for each $x \in$ X and each open set in $\tau_1(\tau_2)$ such that $x \in$ U there exist an open set V in $\tau_1(\tau_2)$ such that $x \in$ V $\subseteq \tau_2(\tau_1)$ - $\bar{V} \subseteq$U, where $\tau_2(\tau_1)$ - $\bar{V}$ means the closure of V in $\tau_2(\tau_1)$.

2.18. Example. Let X={a,b,c}, τ_1={ X,ϕ,{b,c}}and τ_2 ={ X,ϕ,{a}}. Then, we have τ_1 is regular with respect to τ_2 .

2.19. Example. Let X={a,b}, τ_1={ X,ϕ,{a}}and τ_2 ={ X,ϕ,{b}}. Then, the bitopological space (X, τ_1 , τ_2) is P-regular.

2.20. Theorem. Every subspace of P-regular space is P-regular.

2.21. Theorem. Let (X, τ_1 , τ_2) be a P-regular space and P-T_1 space. Then, the space (X, τ_1 , τ_2) is P-T_2 space.

Proof. Let $x,y \in$ X with $x \neq y$. Now since X is P-T_1 space then there exists an open set U in τ_1 with $x \in$ U and $y \notin$ U. Also, being the space is P-regular implies that there exists an open set V in τ_1 such that $x \in$ V $\subseteq \tau_2 - \bar{V} \subseteq$U.

Then, V$\in \tau_1$, $x \in$ V, $\overline{V}^c \in \tau_2$, $y \in \overline{V}^c$ and V$\cap \overline{V}^c = \phi$, thus (X, τ_1 , τ_2) is P_2- space.

3. The main result.

3.1 Definition. An open set U in $\tau_1(\tau_2)$ is said to have H-property if for each

$x \in$ U there exists an open set G in $\tau_1(\tau_2)$ such that $x \in$ G and $\tau_2(\tau_1)$ - $\bar{G} \subset$U .

3.2. Example. Take X = R and define the topologies, τ_1 and τ_2, on X as follows :

τ_1 the cofinite topology and τ_2 the usual topology.

Then U=R – {0} is open in τ_1 and has H-property .

If $0 < x \in$ U, then G = $\left(x - \frac{x}{2}, x + \frac{x}{2}\right)$ is an open set in τ_2 containing x with

τ_2 - $\bar{G}$ = $\left[x - \frac{x}{2}, x + \frac{x}{2}\right] \subseteq$U .

similarly , if $x < 0$, then G = $\left(x - \frac{x}{2}, x + \frac{x}{2}\right)$ is an open set in τ_2 containing x and

τ_2 - $\bar{G}$ = $\left[x - \frac{x}{2}, x + \frac{x}{2}\right] \subseteq$U.

3.3. Definition. Let (X, τ_1 , τ_2) be atopological space. The topology $\tau_1(\tau_2)$ has

H-property if every member of $\tau_1(\tau_2)$ has H-property.

3.4. Example. Take X any non-empty set and define two topologies ,τ_1 and τ_2 , on X as follows: τ_1 any topology and τ_2 the discrete topology.

Then τ_1 has H-property because every member of τ_1 has H-property . If U is any open set in τ_1 and $x \in$ U , then $x \in$ G = $\{x\}$ is open in τ_2 with $\tau_2 \text{-} \bar{G} = \{x\} \subset U$

3.5. Corollary. Let (X, τ_1 , τ_2) be a bitopological space.

If one of the topologies is discrete, then other one has H-property.

3.6. Theorem. Let (X, τ_1 , τ_2) be a bitopological space and (Y, τ_{1y} , τ_{2y}) be a subspace of (X, τ_1 , τ_2). If τ_1 (τ_2) has H-property, then so is τ_{1y} (τ_{2y}).

Proof. Let (X, τ_1 , τ_2) be a bitopological space such that τ_1 has H-property and $Y \subset X$. We must to show τ_{1y} has H-property. Let $U \subset Y$ be any open set in τ_{1y} and

$x \in U_1$, then there exists an open set G in τ_1 such $x \in$ G and τ_1 has H-property then there exists an open set V in τ_2 containing x such that $\tau_2 \text{-} \bar{V} \subset G$. Clearly, $x \in V \cap Y$ and $\tau_2 \text{-} \bar{V} \cap Y \subset G \cap Y$.

Now, $x \in V \cap Y = W$ is open in τ_{2y} and $\tau_{2y} \text{-} \bar{W} = \tau_{2y} \text{-} \overline{V \cap Y} \subset \tau_{2y} \text{-} \bar{V} \cap Y \subseteq G \cap Y = U$.

Therefore, τ_{1y} has H-property.

3.7. Definition. A space (X, τ_1 , τ_2) is said to have H-property if every open set $U \in \tau_1 \cup \tau_2$ has H-property.

3.8. Theorem. Every subspace of a bitopological space which has H-property has H-property.

Proof. It is obvious from the above theorem.

3.9. Theorem. Let (X, τ_1 , τ_2) be a bitopological space which has H-property. If it is P-T_0 space, then (X, τ_1 , τ_2) is P-T_2 space.

Proof. Let $x,y \in X$, $x \neq y$. because (X, τ_1 , τ_2) is P-T_0 space ,so there exists an open set $U \in \tau_1 \cup \tau_2$ (say) $U \in \tau_1$ and $x \in U$, $y \notin U$. The space X has H-property and

$x \in U$, so there exists an open set V in τ_2 such that $x \in V \subseteq \tau_2 \text{-} \bar{V} \subseteq U$.

3.10. Lemma. Let (X, τ_1 , τ_2) be a bitopological space which is P-regular and

$\tau_1 \subseteq \tau_2$, then every open set in τ_1 has H-property.

Proof. Let U be any open set in τ_1 and $x \in$ U. The bitopological space (X, τ_1 , τ_2) is P-regular, so there exist an open set V in τ_1 that is V-open set in τ_2 such that

$x \in V \subseteq \tau_2 \text{-} \bar{V} \subseteq U$. Thus, U has H-property.

3.11. Theorem. Let (X, τ_1 , τ_2) be a bitopological space which has H-property and $\tau_2 \subseteq \tau_1$, then τ_1 is regular with respect to τ_2.

Proof. Let F be closed set in τ_1 and $x \notin$ F, then F^c is open in τ_1 and $x \in F^c$.

Now, since F^c has H-property then there exists an open set G in τ_2 such that $x \in$ G and τ_2 - $\bar{G} \subseteq F^c$. Also, $\tau_2 \subseteq \tau_1$ implies that G is open set in τ_1, $x \in$ G, F $\subseteq \bar{G}^c$ and G ∩ $\bar{G}^c = \phi$, so we have G open set in τ_1 and $\bar{G}^c$open set in τ_2 and $x \in$ G, F $\subseteq \bar{G}^c$ and G ∩ $\overline{G}^c = \phi$ because if $y \in$ G ∩ $\overline{G}^c$, then $y \in$ G and $y \in \overline{G}^c$ and this implies that $y \notin \bar{G}$ and so $y \notin$ G which is a contradiction. Therefore, τ_1 is regular with respect to τ_2 .

The topology τ_3 was introduced by Sunder lal from the bitopological space where τ_3 is the coarsest topology finer than both τ_1 and τ_2.One can easily see that if

(X, τ_1 , τ_2) satisfies P-T_i (i = 0,1,2), then (X, τ_3) is T_i (i = 0,1,2).

3.12. Theorem. Let (X, τ_1 , τ_2) be a bitopological space which is P-regular space and has H-property. Then is (X, τ_3) regular space.

Proof. Let U be any open set in τ_3 and $x \in$ U. We have three cases.

Case1. U is open set in τ_1 and since (X, τ_1 , τ_2) is P-regular then there exists an open set V in τ_1 such that $x \in$V$\subseteq \tau_2$ - $\bar{V} \subseteq$U. Because $\tau_1 \subseteq \tau_3$ and $\tau_2 \subseteq \tau_3$, the set V is open in τ_3 and τ_2 - $\bar{V} \subseteq \tau_3$ - $\bar{V}$. Then we have V open set in τ_3 and $x \in$V$\subseteq \bar{V}$ and so (X, τ_3) is regular space.

Case2. U is open set in τ_2 and since (X, τ_1 , τ_2) is P-regular then there exists an open set G in τ_2 with $x \in$G$\subseteq \tau_1$ - $\bar{G} \subseteq$U. Because of the same argument as in case1, we have G open set in τ_3 and $x \in$G$\subseteq \bar{G} \subseteq$ U and so (X, τ_3) is regular space.

Case3. Because (X, τ_1 , τ_2) is both P-regular and has H-property, each member of τ_3 is the arbitrary union of intersections between open sets of τ_1 and of τ_2. Thus, U= K∪L where K $\in \tau_1$, L $\in \tau_2$ and $x \in$ K or $x \in$ L or x belongs to both K and L. Without loss of generality, $x \in$ K and since (X, τ_1 , τ_2) is P-regular, then there exists an open set M $\in \tau_1$ such that $x \in$M$\subseteq \tau_2$ - $\bar{M} \subseteq$K.

Now, $x \in$M$\subset \tau_2$ - $\bar{M} \subseteq \tau_3$ - $\bar{M} \subseteq$K $\subseteq$ K ∪ L= U. Therefore, (X, τ_3) is regular space.

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