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ON SOLUTIONS OF INITIAL VALUE PROBLEM FOR NONLINEAR FRACTIONAL BERNOULLI EQUATIONS

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الملخص:

في هذه المقالة، ناقشنا طريقة التحليل لـ Adomian decomposition method التي تمَّ تطبيقها لحل معادلة برنولي التفاضلية الكسرية الغير خطية (الخطية) من الدرجة الثانية مع الشروط الأولية. حيث يتم تحويل معادلة برنولي الكسرية إلى معادلة تفاضلية كسرية غير خطية (خطية) تخضع للشروط الأولية. ثم بحثنا عن وجود حلول تقريبية لهذا النوع من مشاكل القيمة الأولية من خلال تطبيق تقنية التحليل ADM، و ذلك من خلال دراسة بعض الأمثلة التوضيحية لتوضيح التقنية المقترحة و معرفة ما إذا كانت الطريقة المقدمة تظهر نتائج ذات كفاءة جيدة أم لا.

Abstract

This research article discusses the Adomian decomposition method that has been applied to solving second-order the nonlinear (linear) fractional differential equation for the Bernoulli equation with initial conditions. Firstly, the Bernoulli equation with fractional derivatives is transferred to a nonlinear (linear) fractional differential equation subject to initial conditions. Then it investigated the existence of approximate solutions to this type of initial value problem by applying Adomian decomposition technique. In view of the convergence of this method, some illustrative examples are included to demonstrate the proposed technique and show the efficiency of the presented method.

Keywords: Fractional differential equation; Adomian decomposition method; Caputo fractional derivative; the Bernoulli differential equation with fractional derivative.

1. Introduction

Since the differential equations with fractional derivatives can describe many important phenomena in electromagnetic, acoustics, viscoelasticity, electrochemistry, cosmology, and material science [4,21,28,36], both professional and academic researchers in various fields have devoted considerable effort to find their explicit solutions. Because of the impossibility of achievement in solving explicit exact solutions for most of these problems, analytical approximate solutions are of academicals and practical importance. Due to the availability of computer symbolic systems like Mathematica or Maple, some fundamental methods have been extended to solve fractional differential equations [22,38,42, 34,35,30,42] and approximate solutions have been found increasingly.

Adomian decomposition method (ADM) [12,14] was firstly proposed by the American mathematician, Adomian and is one of the powerful methods by which the approximate solutions for large classes of nonlinear differential equations can be derived.

In recent decades considerable interest in fractional differential equations (FDE) has been stimulated due to their numerous applications in the areas of physics and engineering [4,16]. Damping laws, diffusion processes [8] and fractals [4] are better formulated with the use of fractional derivatives integrals [15],[16],[25].In addition, Atanackovic and Stankovic [40] have analyzed the lateral motion of an elastic column fixed at one end and loaded at the other, in terms of a system of FDE .

Applying the Adomian decomposition method (ADM) to obtain solutions of several delay differential equations subject to history functions and then investigated numerical examples via subroutines in MAPLE that demonstrate the efficiency of the new approach which was illustrated in[5]. The authors studied the analytical solutions of telegraph equations and fractional partial differential equations were determined using the Laplace-Adomian decomposition method (LADM) [17], and the Adomian Decomposition Method is a semi-analytical method to compute nonlinear second-order differential equations, where this study was introduced by [27]. Wazwaz [2] established a new algorithm for calculating the so-called Adomian polynomials and introduced the modified ADM to solve various differential equations with strong nonlinear terms. Based on Newton method. Abbasbandy [37] presented the modified ADM and applied it to construct the numerical algorithms,in order to overcome inaccurate terms arising from solving nonlinear differential equations with the higher time-derivative. Abassy [39] defined new Adomian polynomials and provided a qualitative improvement over the standard ADM. Song and Wang [31] presented the enhanced ADM, which followed the framework of the standard ADM, introduced the h-curve, established a recursive relationship, and obtained approximate solutions with higher accuracy.

Rawashdeh [32] examined a novel method called the Natural Decomposition Method (NDM) and used it to obtain exact solutions for three different types of nonlinear ordinary differential equations (NLODEs).

Recently, several analytical or numerical methods have been previously proposed to solve fractional differential equations such as the Adomian decomposition method [34,41,15,16, 25,26], and various numerical methods [29,6,9,18], and also you can read other methods in [7,19,20].

The discussion is organized as follows. In the next section, operators of fractional calculus. In Section 3, Analysis of the Adomian Decomposition Method. In Section 4, we consider a class of initial value problems for fractional Bernoulli differential equations with 2^{nd} order and ADM. In Section 5, we discuss four illustrative examples of IVPs for fractional B. DEs. Conclusions are presented in Section 6.

2. Operators of Fractional Calculus

In this section, we describe some necessary definitions and mathematical preliminaries of the fractional calculus theory.

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Definition 1. A real function $f(x), x>0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $\alpha > \mu$, such that $f(x) = x^\alpha f_1(x)$, where $f_1(x) \in C(0,\infty)$, and it is said to be in the space C_m^n if and only if $f^{(n)} \in C_\mu, n \in \mathbb{N}$.

Definition 2. Riemann-Liouville fractional integral operator J^α of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$J^\alpha f(x) = \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \qquad x>0$$

$$J^0 f(x) = f(x) \tag{2.1}$$

where $\Gamma(.)$ is Euler's gamma function, and we have some properties for $f \in C_\mu$ and $\mu \geq -1$ of the operator J^α that can be referred to in the works [21,36,28], which we only recall the following ones:

1- $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

2- $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

3- $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\alpha+\gamma}$

for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0,\ \&\ \gamma > -1$.

Let $f(x)$ is piecewise continuous on $(0,\infty)$ and integrable on any finite subinterval of $(0,\infty)$ and α be a positive real number satisfying $m-1<\alpha\leq m$, $m \in \mathbb{N}^+$. Then the Riemann–Liouville fractional derivative of $f(x)$ of order α, when it exists, is defined as

$$D_x^\alpha f(x) = \frac{d^m}{dx^m}\left(J^{m-\alpha} f(x)\right), \quad x>0 \tag{2.2}$$

Let α be a positive real number, such that $m-1<\alpha\leq m$, $m \in \mathbb{N}^+$ and $f^{(n)}(x)$ exist and be a function of class C .Then the Caputo fractional derivative of $f(x)$ of order α is defined as:

$$D_x^\alpha f(x) = J^{m-\alpha} f^{(m)}(x), \quad x>0 \tag{2.3}$$

for the Caputo fractional derivative of a polynomial function, the following equality holds

$$D_x^\alpha (a_0 x^r + a_{r-1} x^{r-1} + \ldots + a_r) = 0, \quad m-1<\alpha\leq m, \ r \leq m-1 \tag{2.4}$$

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Moreover, the α-order integral of the α-order Caputo fractional derivative satisfies

$$J_x^\alpha D_x^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad m-1 < \alpha \le m \tag{2.5}$$

For the power function $x^\beta, \beta > 0$ if $0 \le m-1 < \alpha < m$ then we have:

$$D_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad x > 0 \tag{2.6}$$

The Caputo fractional derivative and the Riemann–Liouville fractional derivative satisfy the following relation[44]:

$$D_x^\alpha f(x) = D_x^\alpha \left[f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!} \right], \tag{2.7}$$

3. Analysis of the Adomian Decomposition method

Let us first recall the basic principles of the ADM using an IVP for a nonlinear ODE in the form

$$Lu + Ru + Nu = g(x) \tag{3.1}$$

where g is a known analytical function, and where L is the linear operator to be inverted, which usually is just the highest order differential operator, R is the linear remainder part, and N is the nonlinear operator, which is assumed to be analytic. Furthermore, we choose $L = \frac{d^n}{dx^n}(.)$ for $n^{th}-$order differential equations and thus its inverse L^{-1} follows as the $n-$fold definite integration operator from x_0 to x, we have $L^{-1}Lu = u - \Phi$, where Φ incorporates the initial values as $\Phi = \sum_{k=0}^{p-1} \beta_i \frac{(x_0 - x)^k}{k!}$.

Applying the inverse linear operator L^{-1} to both sides of Eq. (3.1) gives

$$u = \Phi + L^{-1}g(x) - L^{-1}[Ru + Nu] \tag{3.2}$$

The Adomian decomposition method decomposes the solution into a series:

$$u = \sum_{n=0}^{\infty} u_n \tag{3.3}$$

and then decomposes the nonlinear term Nu into a series:

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$$Nu = \sum_{n=0}^{\infty} A_n \qquad (3.4)$$

where A_n depending on $u_0, u_1, \ldots, u_n$ are called the Adomian polynomials and are obtained for the nonlinearity $Nu = f(u)$ by the definitional formula[11]

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[f\left(\sum_{k=0}^{\infty} u_k \lambda^k \right)\right]_{\lambda=0}, n = 0,1,\ldots \qquad (3.5)$$

where λ is simply a grouping parameter of convenience.

We list the formulas of the first several Adomian polynomials for the one-variable simple analytic nonlinearity $Nu = f(u)$ from A_0 through A_5, inclusively, for convenient reference as

$$A_0 = f(u_0)$$
$$A_1 = f'(u_0)u_1$$
$$A_2 = f'(u_0)u_2 + f''(u_0)\frac{u_1^2}{2!}$$
$$A_3 = f'(u_0)u_3 + f''(u_0)u_1u_2 + f^{(3)}(u_0)\frac{u_1^3}{3!}$$
$$A_4 = f'(u_0)u_4 + f''(u_0)\left(\frac{u_2^2}{2!} + u_1u_3\right) + f^{(3)}(u_0)u\frac{u_1^2u_2}{2!} + f^{(4)}(u_0)\frac{u_1^4}{4!}$$
$$A_5 = f'(u_0)u_5 + f''(u_0)\left(u_2u_3 + u_1u_4\right) + f^{(3)}(u_0)\left(\frac{u_1u_2^2 + u_1^2u_3}{2!}\right) + f^{(4)}(u_0)\frac{u_1^3u_2}{3!} + f^{(5)}(u_0)\frac{u_1^5}{5!}$$

Upon substitution of the Adomian decomposition series for the solution $u(x)$ and the series of Adomian polynomials tailored to the nonlinearity Nu from Eqs. (3.3) & (3.4) into Eq. (3.2), we have

$$\sum_{n=0}^{\infty} u_n = \gamma(x) - L^{-1}\left[R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n \right] \qquad (3.6)$$

The solution components $u_n(x)$ may be determined by one of several advantageous recursion schemes, which differ from one another by the choice of the initial solution component $u_0(x)$, beginning with the classic Adomian recursion scheme

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$$u_0(x) = \gamma(x)$$
$$u_{n+1}(x) = -L^{-1}\left[R\sum_{n=0}^{\infty}u_n(x) + \sum_{n=0}^{\infty}A_n\right], n \geq 0 \qquad (3.7)$$

where Adomian has chosen the initial solution component as $u_0(x) = \gamma(x)$ .The n-term approximation of the solution is

$$\varphi_n(x) = \sum_{k=0}^{n-1}u_k(x) \quad for\ n > 0 \qquad (3.8)$$

Thus $\varphi_1 = u_0, \varphi_2 = \varphi_1 + u_1, \varphi_3 = \varphi_2 + u_2, etc.$ serve as approximate solutions of increasing accuracy as n increases and must of course satisfy the boundary conditions [13], the remarkable measure of success of the ADM is demonstrated by its widespread adoption and many adaptations to enhance computability for specific purposes, such as the various modified recursion schemes. The choice of decomposition is not unique, which provides a valuable advantage to the analyst, permitting the freedom to design modified recursion schemes for ease of computation in realistic systems.

4. IVP for nonlinear fractional Bernoulli Differential Eqs.

In this section, we consider the initial value problem for the second-order nonlinear fractional equation for Bernoulli differential equation [22], which can be written in the form:

$$P(x)D^2y + R(x)D^{\alpha}y + Q(x)Dy + S(x)y = m\,P(x)\frac{y'^2}{y} + \frac{R(x)}{\Gamma(1-\alpha)x^{\alpha}}y + f(x)y^m \qquad (4.1)$$

subject to the following initial conditions: $y(a) = \mu_0,\ y'(a) = \mu_1,\ y''(a) = \mu_2$, where: $P(x) \neq 0, Q(x) \neq 0, m \geq 2$ and also μ_0, μ_1 are not equal to zero, where $0 < \alpha \leq 2$.

To find solution for this type of differential equations, we shall transform the Bernoulli's equation (4.1) to the linear equation by the transformation $u = y^{1-m}$, and hence (4.1) will becomes as (see[33]):

$$\frac{1}{1-m}\left(P(x)\frac{d^2u}{dx^2} + R(x)D^{\alpha}u + Q(x)\frac{du}{dx}\right) + S(x)u = f(x) \qquad (4.2)$$

subject to the initial conditions: $u(a) = y^{1-m}(a) = \mu_0^{1-m} = c_0,\ Du(a) = D\left(y^{1-m}(a)\right) = c_1,$ $D^2u(a) = D^2\left(y^{1-m}(a)\right) = c_3$.

To find numerical solution for initial value problem (4.2),(4,3) by the ADM, firstly, can be written the problem in operator form as:

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$$Lu=\frac{1-m}{P(x)}\left(f(x)-S(x)u\right)-\frac{1}{P(x)}\left(R(x)D^{\alpha}u+Q(x)\frac{du}{dx}\right) \quad (4.3)$$

where $L=\frac{d^2}{dx^2}$ is linear operator, with the inverse operator defined as $L^{-1}=\int_a^x\int_a^x(.)dxdx$.

Applying the inverse operator L^{-1} on both sides of equation (4.3) and imposing the initial conditions yield:

$$u=c_0+(x+1)c_1+\frac{1-m}{P(x)}L^{-1}\left(f(x)-S(x)u\right)-\frac{1}{P(x)}L^{-1}\left(R(x)D^{\alpha}u+Q(x)\frac{du}{dx}\right) \quad (4.4)$$

where $u(a)=c_0,\ Du(a)=c_1$. In [13], the ADM assumes a series solution for $u(x)$ given by an infinite sum of components:

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \quad (4.5)$$

and then decompose the analytic nonlinearity F into the series of the Adomian polynomials as form:

$$F_1u=F_1\left(\sum_{n=0}^{\infty}u_n\right)=\sum_{n=0}^{\infty}A_n,\qquad F_2u=\sum_{n=0}^{\infty}B_n \quad (4.6)$$

where A_n, B_n are the Adomian polynomials given by:

$$A_n=\frac{1}{n!}\left[\frac{\partial^n}{\partial\lambda^n}F_1\left(\sum_{k=0}^{\infty}u_k\lambda^k\right)\right]_{\lambda=0},\quad B_n=\frac{1}{n!}\left[\frac{\partial^n}{\partial\lambda^n}F_2\left(\sum_{k=0}^{\infty}u_k\lambda^k\right)\right]_{\lambda=0}$$

where λ is simply a grouping parameter of convenience.

Substituting the decompositions of the solution and the nonlinear term into eq. (4.1) yields:

$$\sum_{n=0}^{\infty}u_n=c_0+(x+1)c_1+\frac{1-m}{P(x)}L^{-1}\left(f(x)-\sum_{n=0}^{\infty}A_n\right)-\frac{1}{P(x)}L^{-1}\left(\sum_{n=0}^{\infty}B_n\right) \quad (4.7)$$

from which we obtain the recursion scheme for the solution components,

$$\begin{cases} u_0=c_0+(x+1)c_1+\frac{1-m}{P(x)}L^{-1}f(x) \\ u_{n+1}=\frac{1-m}{P(x)}L^{-1}\left(S(x)u_n\right)-\frac{1-m}{P(x)}L^{-1}A_n-\frac{1}{P(x)}L^{-1}B_n,\ n\geq 0 \end{cases} \quad (4.8)$$

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To compute this formula is easy by using mathematical software or Maple program to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions. Finally, we approximate the solution $y(x)$ by the truncated series as:

$$\varphi_n(x) = \sum_{k=0}^{n-1} y_k(x) \quad , \lim_{n\to\infty} \varphi_n(x) = y(x) \qquad (4.9)$$

5. Illustrative examples

In this section, we present the examples for the nonlinear fractional Bernoulli equations with initial conditions investigated to show the efficiency of the method described in the previous section.

Example 1. Consider the following nonlinear fractional Bernoulli equation with initial conditions:

$$y'' + y' + D^{\frac{1}{2}} y = 2\frac{y'^2}{y} + \frac{y}{\Gamma(\frac{1}{2})\sqrt{x}} + \left(\frac{2\sqrt{x}}{\sqrt{\pi}} - x - \frac{4x^{\frac{3}{2}}}{3\sqrt{\pi}}\right) y^2 \qquad (5.1)$$

$$y(1) = -2, y'(1) = 0, y''(1) = -4.$$

with exact solution $y(x) = \frac{2}{x^2 - 2x}, \forall x \in [1,3]$, and note that the exact solution isn't continuous function at $x = 2$

To find approximate solution for nonlinear equation (5.1) by applying ADM method, first, we reduce the Bernoulli equation to the linear equation by the transformation $u = y^{-1}$ therefore the equation (5.1) will become to:

$$\frac{d^2u}{dx^2} + \frac{du}{dx} + D^{\frac{1}{2}} u = -2\sqrt{\frac{x}{\pi}} + x + \frac{4}{3}\sqrt{\frac{x^3}{\pi}} \qquad (5.2)$$

Subject to the initial conditions: $u(1) = \frac{-1}{2},\ u'(1) = 0, u''(1) = 1$.

Consequently, the series solution for $u(x)$ by using the ADM described in [13] for this equation given by: $u(x) = \sum_{n=0}^{\infty} u_n(x)$

$$u = u(1) + (x-1)u'(1) + L^{-1}\left(-2\sqrt{\frac{x}{\pi}} + x + \frac{4}{3}\sqrt{\frac{x^3}{\pi}}\right) - L^{-1}\left(\frac{du}{dx}\right) - L^{-1}\left(D^{\frac{1}{2}} u\right) \qquad (5.3)$$

Accordingly, the iteration formula for the (5.2) equation is given by

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$$u_{n+1}=u(1)+(x-1)u'(1)+\int_1^x\int_1^x\left(-2\sqrt{\frac{x}{\pi}}+x+\frac{4}{3}\sqrt{\frac{x^3}{\pi}}-D^{\frac{1}{2}}u_n(x)-\frac{du_n(x)}{dx}\right)dxdx$$

From this equation and by using initial conditions for Eq.(5.2),the iterates are determined by the following recursive way:

$$u_0=\frac{-1}{2}+\int_1^x\int_1^x\left(-2\sqrt{\frac{x}{\pi}}+x+\frac{4}{3}\sqrt{\frac{x^3}{\pi}}\right)dxdx$$

$$u_1=-\int_1^x\int_1^x\left(\frac{du_0(x)}{dx}-D^{\frac{1}{2}}u_0(x)\right)dxdx \tag{5.4}$$

$$\vdots$$

$$u_{n+1}=-\int_1^x\int_1^x\left(\frac{du_n(x)}{dx}-D^{\frac{1}{2}}u_n(x)\right)dxdx$$

Then, we can find the previous integral, hence, the first two terms of the ADM series solution, are as follows:

$$u_0=\frac{-1}{6}-\frac{44}{105\sqrt{\pi}}+\left(\frac{4}{5\sqrt{\pi}}-\frac{1}{2}\right)x-\frac{8x^{\frac{5}{2}}}{15\sqrt{\pi}}+\frac{x^3}{6}+\frac{16x^{\frac{7}{2}}}{105\sqrt{\pi}}$$

$$u_1=\int_1^x\int_1^x\left(D^{\frac{1}{2}}(\frac{-1}{6}-\frac{44}{105\sqrt{\pi}}+\left(\frac{4}{5\sqrt{\pi}}-\frac{1}{2}\right)x-\frac{8x^{\frac{5}{2}}}{15\sqrt{\pi}}+\frac{x^3}{6}+\frac{16x^{\frac{7}{2}}}{105\sqrt{\pi}})\right)dxdx-$$
$$-\int_1^x\int_1^x\left(\frac{d}{dx}(\frac{-1}{6}-\frac{44}{105\sqrt{\pi}}+\left(\frac{4}{5\sqrt{\pi}}-\frac{1}{2}\right)x-\frac{8x^{\frac{5}{2}}}{15\sqrt{\pi}}+\frac{x^3}{6}+\frac{16x^{\frac{7}{2}}}{105\sqrt{\pi}})\right)dxdx$$

$$\therefore\ u_1=\frac{13}{60}-\frac{568}{1575\pi}+\frac{241}{945\sqrt{\pi}}+\left(\frac{8}{35\pi}-\frac{3}{7\sqrt{\pi}}-\frac{11}{24}\right)x+\left(\frac{176}{315\pi}+\frac{2}{9\sqrt{\pi}}\right)x^{\frac{3}{2}}+$$
$$+\left(\frac{1}{4}-\frac{2}{5\sqrt{\pi}}\right)x^2+\left(\frac{-32}{75\pi}+\frac{4}{15\sqrt{\pi}}\right)x^{\frac{5}{2}}+\frac{16x^{\frac{7}{2}}}{105\sqrt{\pi}}-\frac{64x^{\frac{9}{2}}}{945\sqrt{\pi}}-\frac{x^5}{120}\quad .$$

and

$$u_2=\frac{49}{720}+\frac{272}{4725\pi^{\frac{3}{2}}}+\frac{674}{2025\pi}+\frac{457}{5005\sqrt{\pi}}+\left(\frac{-656}{1575\pi^{\frac{3}{2}}}-\frac{86}{675\pi}-\frac{37}{385\sqrt{\pi}}-\frac{1}{40}\right)x+$$
$$+\left(\frac{2272}{4725\pi^{\frac{3}{2}}}-\frac{964}{2835\pi}-\frac{13}{45\sqrt{\pi}}\right)x^{\frac{3}{2}}+\left(\frac{11}{48}-\frac{4}{35\pi}+\frac{3}{14\sqrt{\pi}}\right)x^2+$$

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$$+\left(\frac{-64}{525\pi^{\frac{3}{2}}}+\frac{8}{1575\pi}+\frac{7}{45\sqrt{\pi}}\right)x^{\frac{5}{2}}+\left(-\frac{1}{9}+\frac{4}{63\sqrt{\pi}}\right)x^{3}+\left(\frac{128}{525\pi}-\frac{16}{105\sqrt{\pi}}\right)x^{\frac{7}{2}}+$$
$$+\left(\frac{-11}{48}+\frac{1}{30\sqrt{\pi}}\right)x^{4}-\frac{32x^{\frac{9}{2}}}{945\sqrt{\pi}}-\frac{x^{5}}{120}+\frac{128x^{\frac{11}{2}}}{10395\sqrt{\pi}}+\frac{x^{6}}{240}+\frac{128x^{\frac{13}{2}}}{135135\sqrt{\pi}}\quad .$$
$$\vdots$$

Hence, the ADM series solution of the initial value problem (5.2) can be given by:

$$u(x)=u_0(x)+u_1(x)+u_2(x)+u_3(x)+.... \qquad (5.5)$$

Table1 shows the approximate solution $y(x) \approx \sum_{n=0}^{6} y_n(x)$ of the fractional differential equation (5.1) and the approximate solution $u(x) \approx \sum_{n=0}^{6} u_n(x)$ of the differential equation (5.2) obtained by using the ADM method. It is to be note that only the sum of the first six iterations was used in evaluating the approximate solution for Fig. 1, where we note from the graphical results in Fig. 1, it is clear that the approximate solution is in agreement with the exact solution present.

x	Exact solution	ADM	Absolute value	Exact solution	ADM	Absolute value
	$y(x_i)$	y_n	$\lvert y-y_n\rvert$	$u(x_i)$	$u_n(x_i)$	$\lvert u-u_n\rvert$
1.0	-2.00	-1.99999	8.881784×10^{-16}	-0.5	-0.5	0.0
1.2	-2.0833333	-2.08332	0.0000141839	-0.48	-0.4800033	3.2680031×10^{-6}
1.4	-2.3809524	-2.3809	0.0000506202	-0.42	-0.4200089	8.9295924×10^{-6}
1.6	-3.125	-3.12486	0.000138471	-0.32	-0.3200141	0.00001418075
1.8	-5.555556	-5.55493	0.000621014	-0.18	-0.1800201	0.00002012312
2.0	ComplexInfinity	-28847.8	ComplexInfinity	0.00	-0.0000347	0.00003466474
2.2	4.5454545	4.54705	0.00159875	0.22	0.2199226	0.00077352499
2.4	2.0833333	2.08407	0.000737967	0.48	0.47983003	0.00016996746
2.6	1.28205128	1.28249	0.000433945	0.78	0.7797361	0.00026392268
2.8	0.892857143	0.892852	4.88701×10^{-6}	1.12	1.1200061	6.1302945×10^{-6}
3.0	0.666666667	0.665799	0.00867903	1.5	1.5019655	0.0019553275

Table 1.The absolute errors of example (1) between the approximate values by ADM and the exact solution

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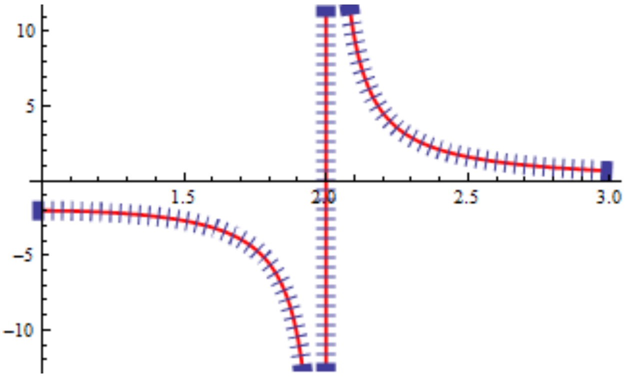
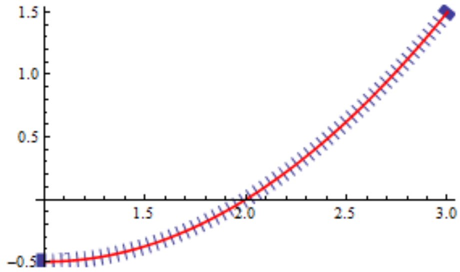
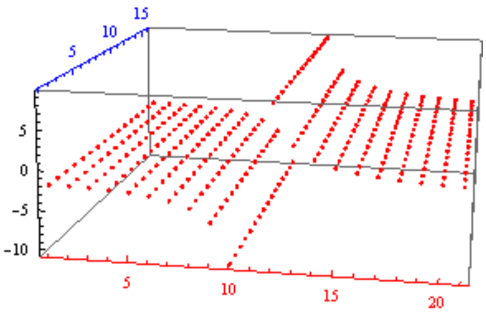


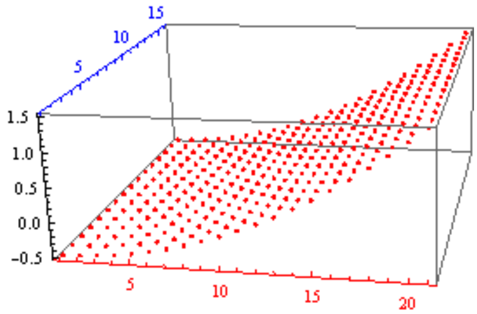


Exact sol. for u
ADM for u_n

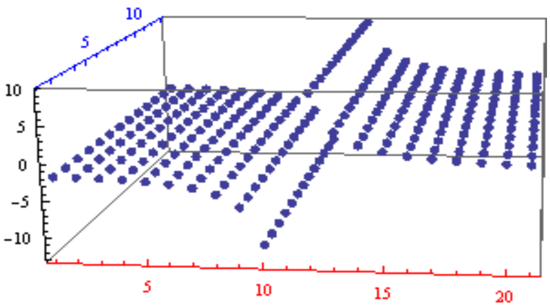
Exact Sol. for y
ADM for y_n



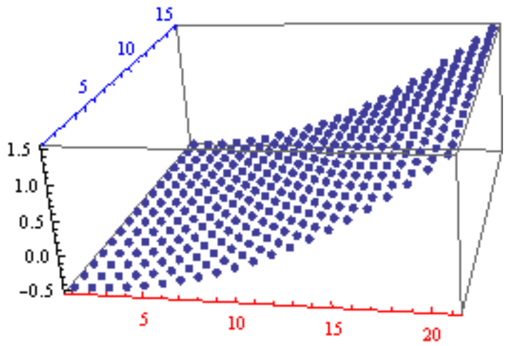
Exact Sol. for y



Exact Sol. for u



ADM for y_n



ADM for u_n

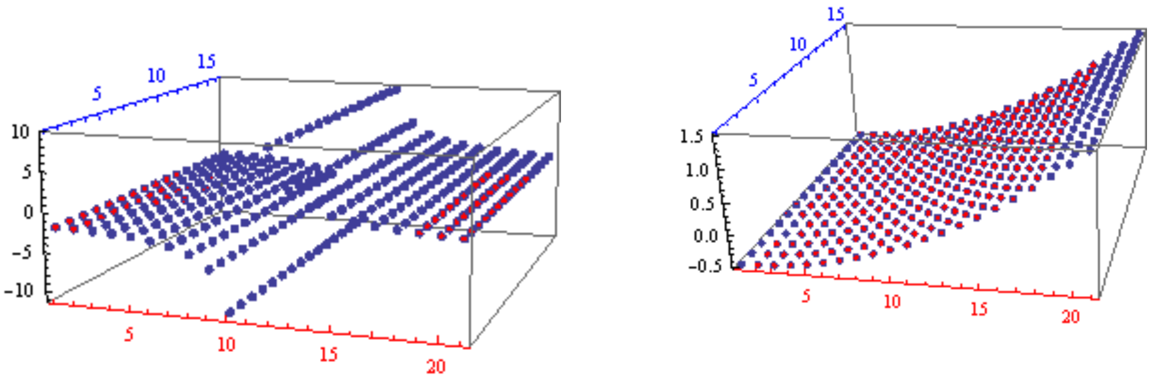


Figure 1. comparing between approximate solution and exact solution

The table1 indicates that as we get closer and closer to 2, the speed of oscillation of the curve of the approximate solution of $y(x)$ increases between $+\infty$ and $-\infty$. In order to illustrate of the problem, the reason for this is that when x approaches 2 from the right-hand limit, the value of the approximate solution will approach $+\infty$, and as x approaches 2 from the left-hand limit, the solution will approach $-\infty$. This means that the approximate solution of y is increasing in the period $[2,3]$, and also is decreasing in the period $[1,2]$. In addition, as we get closer and closer to 2, the rate of increase in the value of the approximate solution will increase. So this is why the approximate solution curve fluctuates rapidly between $+\infty$ and $-\infty$.

Figure1reveal that, if the values of the approximate solution of y are infinitely increased or decreased when x approaches 2 from the right or left sides and becomes infinitely discontinuous.

Example 2. Consider the following nonlinear fractional Bernoulli differential equation

$$y''+y'-xD^{\frac{1}{3}}y=2\frac{y'^2}{y}-\frac{x^{\frac{2}{3}}y}{\Gamma(\frac{2}{3})}+\left(\frac{\sqrt{\pi}}{12}-6x+\frac{\sqrt{\pi}x}{12}-3x^2-\frac{\sqrt{\pi}x^{\frac{8}{3}}}{12\Gamma(\frac{8}{3})}+\frac{6x^{\frac{11}{3}}}{\Gamma(\frac{11}{3})}\right)y^2 \qquad (5.6)$$

subject to initial conditions:

$$y(1)=\frac{24}{24-\sqrt{\pi}},\; y'(1)=-\frac{48\left(36-\sqrt{\pi}\right)}{\left(24-\sqrt{\pi}\right)^2},\; y''(1)=\frac{165888-9216\sqrt{\pi}+144\pi}{\left(24-\sqrt{\pi}\right)^3}$$

with exact solution $y(x)=\dfrac{1}{x^3-\frac{\sqrt{\pi}}{24}x^2}$ for all $x\in[1,2]$.

To find approximate solution for nonlinear equation (5.6) by applying ADM method , first, we reduce the Bernoulli equation to the linear equation by the transformation $u=y^{-1}$, therefore the equation (5.6) will become to:

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$$\frac{d^2u}{dx^2} - xD^{\frac{1}{3}}u + \frac{du}{dx} = \frac{-\sqrt{\pi}}{12}(x+1) + 6x + 3x^2 + \frac{\sqrt{\pi}x^{\frac{8}{3}}}{12\Gamma(\frac{8}{3})} - \frac{6x^{\frac{11}{3}}}{\Gamma(\frac{11}{3})} \qquad (5.7)$$

Subject to the initial conditions: $u(1) = 1 - \frac{\sqrt{\pi}}{24},\ u'(1) = 3 - \frac{\sqrt{\pi}}{12}, u''(1) = 6 - \frac{\sqrt{\pi}}{12}$.

Consequently, the series solution for $u(x)$ by using the ADM described in [13] for this equation given by $u(x) = \sum_{n=0}^{\infty} u_n(x)$, therefore we have

$$u = u(1) + (x-1)u'(1) + L^{-1}\left(\frac{-\sqrt{\pi}}{12}(x+1) + 6x + 3x^2 + \frac{\sqrt{\pi}x^{\frac{8}{3}}}{12\Gamma(\frac{8}{3})} - \frac{6x^{\frac{11}{3}}}{\Gamma(\frac{11}{3})}\right) + L^{-1}\left(xD^{\frac{1}{3}}u\right) - L^{-1}\left(\frac{du}{dx}\right) \qquad (5.8)$$

Accordingly, the iteration formula for the (5.2) equation is given by

$$u_{n+1} = u(1) + (x-1)u'(1) + \int_1^x \int_1^x \left(\frac{-\sqrt{\pi}}{12}(x+1) + 6x + 3x^2 + \frac{\sqrt{\pi}x^{\frac{8}{3}}}{12\Gamma(\frac{8}{3})} - \frac{6x^{\frac{11}{3}}}{\Gamma(\frac{11}{3})} + xD^{\frac{1}{3}}u_n(x) - \frac{du_n(x)}{dx}\right)dxdx$$

From this equation and by using initial conditions for Eq.(5.6),the iterates are determined by the following recursive way:

$$\begin{aligned}
u_0 &= 1 - \frac{\sqrt{\pi}}{24} + (x-1)(3 - \frac{\sqrt{\pi}}{12}) + \int_1^x\int_1^x \left(\frac{\sqrt{\pi}}{12}(x-1) + 6x + 3x^2 + \frac{\sqrt{\pi}x^{\frac{8}{3}}}{12\Gamma(\frac{8}{3})} - \frac{6x^{\frac{11}{3}}}{\Gamma(\frac{11}{3})}\right)dxdx \\
u_1 &= \int_1^x \int_1^x \left(xD^{\frac{1}{3}}u_1(x) - \frac{du_1(x)}{dx}\right)dxdx \qquad (5.9) \\
&\vdots \\
u_{n+1} &= \int_1^x \int_1^x \left(xD^{\frac{1}{3}}u_n(x) - \frac{du_n(x)}{dx}\right)dxdx
\end{aligned}$$

Hence , the first two terms of the ADM series solution are as follows:

$$\begin{aligned}
u_0 = \frac{3}{4} - \frac{\sqrt{\pi}}{36} - \frac{27}{68\Gamma\frac{8}{3}} + \frac{\sqrt{\pi}}{56\Gamma\frac{8}{3}} + \left(-1 + \frac{\sqrt{\pi}}{24} + \frac{27}{56\Gamma\frac{8}{3}} - \frac{\sqrt{\pi}}{44\Gamma\frac{8}{3}}\right)x - \\
-\frac{\sqrt{\pi}x^2}{24}\left(1 - \frac{\sqrt{\pi}}{72}\right)x^3 + \frac{x^4}{4} - \frac{3\sqrt{\pi}x^{\frac{14}{3}}}{616\Gamma\frac{8}{3}} - \frac{81x^{\frac{17}{3}}}{952\Gamma\frac{8}{3}}
\end{aligned}$$

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$$
\begin{aligned}
u_1 = & -\frac{27}{304\Gamma\frac{16}{3}}x^{\frac{22}{3}} + \frac{39\sqrt{3}\pi}{340\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2}x^{\frac{20}{3}} + \frac{\sqrt{\pi}}{152\Gamma\frac{16}{3}}x^{\frac{19}{3}} + \left(\frac{39\sqrt{3}\pi}{187\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2} - \frac{13\pi^{\frac{3}{2}}}{561\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2}\right)x^{\frac{17}{3}} + \\
& \frac{364\pi}{243\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})}x^5 + \left(\frac{13\sqrt{3}\pi}{11\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2} - \frac{13\pi^{\frac{3}{2}}}{264\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2}\right)x^{\frac{14}{3}} + \left(\frac{-1820\pi}{243\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})} + \frac{455\pi^{\frac{3}{2}}}{4374\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})}\right)x^4 - \\
& -\frac{91\pi^{\frac{3}{2}}}{396\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2}x^{\frac{11}{3}} + \left(\frac{6561}{44800(\Gamma\frac{2}{3})^2} - \frac{243\sqrt{\pi}}{35200(\Gamma\frac{2}{3})^2} - \frac{91\pi}{\sqrt{3}(\Gamma-\frac{1}{3})^2\Gamma\frac{16}{3}} + \frac{91\pi^{\frac{3}{2}}}{216\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2}\right)x^{\frac{8}{3}} + \\
& +\left(\frac{2187}{6800(\Gamma\frac{2}{3})^2} + \frac{81\sqrt{\pi}}{5600(\Gamma\frac{2}{3})^2} + \frac{182\pi}{\sqrt{3}(\Gamma-\frac{1}{3})^2\Gamma\frac{16}{3}} - \frac{182\pi^{\frac{3}{2}}}{243\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2}\right)x^{\frac{5}{3}} + \frac{910\pi^{\frac{3}{2}}}{2187\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})}x^3 + \\
& +\left(\frac{-13\pi}{2\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2} + \frac{91\pi^{\frac{3}{2}}}{297\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2} + \frac{3640\pi}{243\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})} - \frac{455\pi^{\frac{3}{2}}}{729\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})}\right)x^2 + \\
& +\left(\frac{13851}{95200(\Gamma\frac{2}{3})^2} - \frac{351\sqrt{\pi}}{61600(\Gamma\frac{2}{3})^2} + \frac{99}{152\Gamma\frac{16}{3}} - \frac{\sqrt{\pi}}{24\Gamma\frac{16}{3}} - \frac{81419\pi}{5049\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2} + \frac{11479\pi^{\frac{3}{2}}}{16038\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2} + \right. \\
& \left. + \frac{1820\pi}{243\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})} - \frac{910\pi^{\frac{3}{2}}}{2187\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})}\right)x - \frac{3}{\Gamma\frac{19}{3}} + \frac{22599}{761600(\Gamma\frac{2}{3})^2} - \frac{459\sqrt{\pi}}{246400(\Gamma\frac{2}{3})^2} + \frac{2\sqrt{\pi}}{57\Gamma\frac{16}{3}} + \\
& + \frac{807781\pi}{100980\sqrt{3}(\Gamma-\frac{1}{3})^2\Gamma\frac{16}{3}} - \frac{213889\pi^{\frac{3}{2}}}{545292\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})^2} - \frac{364\pi}{27\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})} + \frac{2275\pi^{\frac{3}{2}}}{4374\sqrt{3}\Gamma\frac{16}{3}(\Gamma\frac{2}{3})} \\
& \vdots
\end{aligned}
$$

Table 2 shows the approximate solution $y(x) \approx \sum_{n=0}^{2} y_n(x)$ for fractional equation (5.6) and the approximate solution $u(x) \approx \sum_{n=0}^{2} u_n(x)$ for differential equation (5.7) obtained by using ADM method have been plotted in Figure 2, where we note from the graphical results in Fig. 2, the approximate solution is in agreement with the exact solution present.

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Table 2.The absolute errors of eq. (5.6) between the approximate values by ADM and the exact solution

	Exact solution	ADM	Absolute value	Exact solution	ADM	Absolute value
x_i	$y(x_i)$	y_n	$\|y-y_n\|$	$u(x_i)$	$u_n(x_i)$	$\|u-u_n\|$
1.0	1.07974	1.07974	2.22045×10^{-16}	0.92614	0.926148	2.22045×10^{-16}
1.1	0.805387	0.805603	0.000215296	1.24164	1.24131	0.000331826
1.2	0.616655	0.617607	0.000952025	1.62165	1.61915	0.00249974
1.3	0.482581	0.484676	0.00209471	2.07219	2.06323	0.00895575
1.4	0384726	0.388206	0.00347957	2.59925	2.57595	0.0232976
1.5	0.31164	0.316629	0.00498875	3.20883	3.15827	0.0505579
1.6	0.255955	0.262506	0.00655134	3.90694	3.80943	0.0975051
1.7	0.212786	0.220916	0.00813003	4.69957	4.52662	0.172951
1.8	0.178804	0.188514	0.00970963	5.59272	5.30466	0.28806
1.9	0.15169	0.162979	0.0112894	6.59239	6.13574	0.456649
2.00	0.129793	0.142671	0.0128787	7.70459	7.00911	0.69548

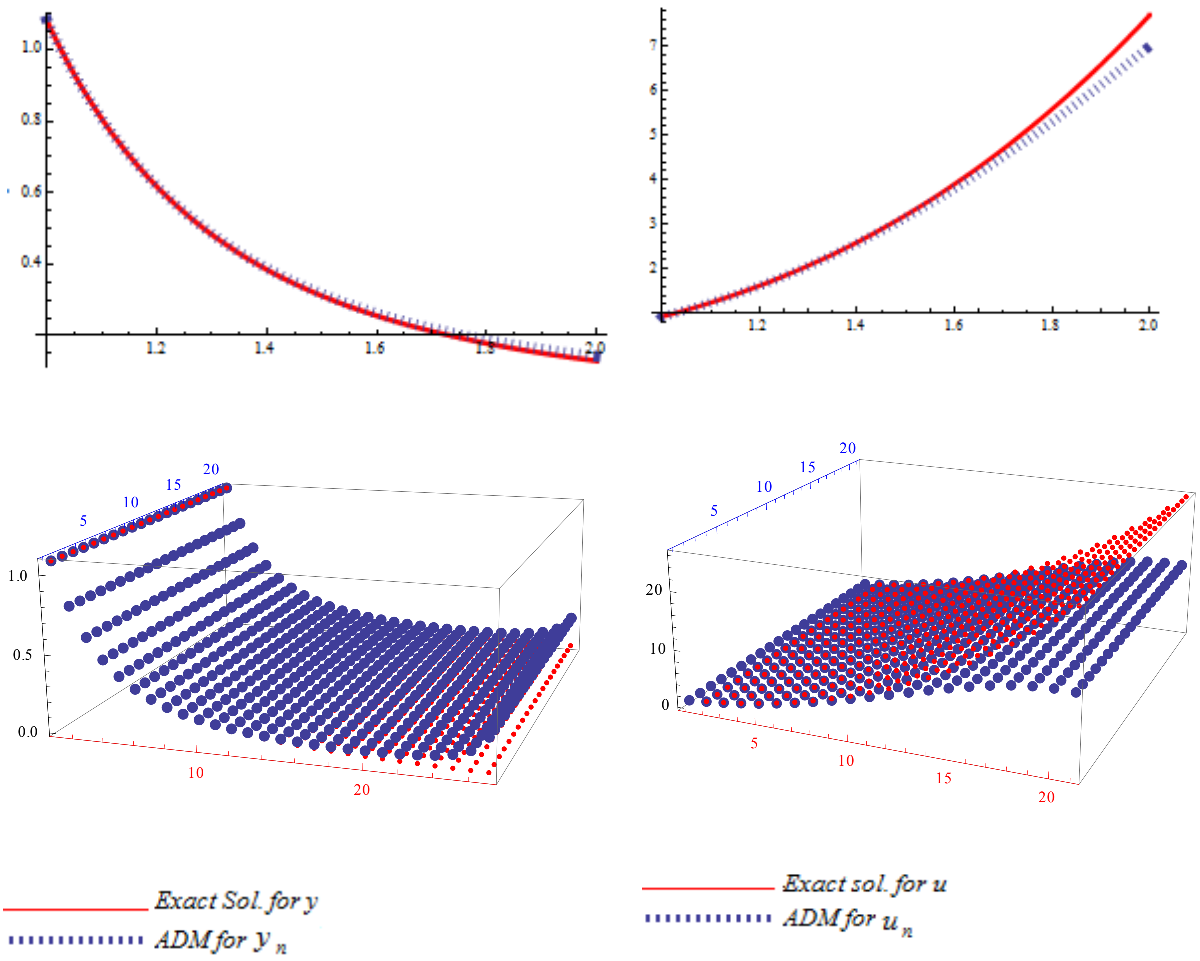

Figure 2. comparing between approximate solution and exact solution

Example 3. Consider the nonlinear fractional Bernoulli equation with initial conditions:

$$y''+y'-D^{\frac{1}{5}}y+y=4\frac{y'^2}{y}-\frac{y}{\Gamma(\frac{4}{5})\,x^{\frac{1}{5}}}+\left(-24x^7-3x^8+x^9+\frac{6}{\Gamma(\frac{19}{5})}x^{\frac{44}{5}}\right)y^4 \qquad (5.10)$$

subject to initial conditions: $y(1)=1, y'(1)=-3, y''(1)=12$, with exact solution $y(x)=x^{-3}$.

To find approximate solution for nonlinear equation (5.10) by applying ADM method, first, we reduce the Bernoulli equation to the linear equation by the transformation $u=y^{-3}$, therefore the equation (5.10) will become to:

$$\frac{d^2u}{dx^2}+\frac{du}{dx}-D^{\frac{1}{5}}u-3u=72x^7+9x^8-3x^9-\frac{6}{\Gamma(\frac{19}{5})}x^{\frac{44}{5}} \qquad (5.11)$$

Subject to the initial conditions: $u(1)=1,\ u'(1)=9, u''(1)=72$. Consequently, the series solution for $u(x)$ by using the ADM described in [11,12,13] for this equation given by $u(x)=\sum_{n=0}^{\infty}u_n(x)$, therefore we have:

$$u=u(1)+(x-1)u'(1)+L^{-1}\left(72x^7+9x^8-3x^9-\frac{6}{\Gamma(\frac{19}{5})}x^{\frac{44}{5}}\right)+L^{-1}\left(D^{\frac{1}{5}}u\right)-L^{-1}\left(\frac{du}{dx}\right)+3L^{-1}u \qquad (5.12)$$

Consequently, we obtained the results approximate solutions for eq.(5.12) through the iteration formula of this equation which is given by:

$$u_{n+1}=-8+x+\int_1^x\int_1^x\left(72x^7+9x^8-3x^9-\frac{6}{\Gamma(\frac{19}{5})}x^{\frac{44}{5}}\right)dxdx+\int_1^x\int_1^x\left(D^{\frac{1}{5}}u_n-\frac{du_n}{dx}+3u_n\right)dxdx \qquad (5.13)$$

Hence, the following table 3 shows approximate solution $y(x) \approx \sum_{n=0}^{2}y_n(x)$ for fractional equation (5.10) and approximate solution $u(x) \approx \sum_{n=0}^{2}u_n(x)$ for differential equation (5.11) obtained by using ADM method have been plotted in Figure 3.

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Table 3. The absolute errors of eq. (5.10) between the approximate values by ADM and the exact solution

x_i	Exact solution	ADM	Absolute value	Exact solution	ADM	Absolute value
	$y(x_i)$	y_n	$\|y-y_n\|$	$u(x_i)$	$u_n(x_i)$	$\|u-u_n\|$
1.0	1.00	1.00	0.00	1.00	1.00	0.00
1.1	0.424098	0.425858	0.00176066	2.35795	2.3482	0.00974866
1.2	0.193807	0.195561	0.00175446	5.15978	5.11349	0.0462905
1.3	0.0942996	0.0954497	0.00115009	10.6045	10.4767	0.127775
1.4	0.0484003	0.0490855	0.000685239	20.661	20.3726	0.28843
1.5	0.0260123	0.0264186	0.000406317	38.4434	37.8521	0.591257
1.6	0.0145519	0.0147993	0.000247394	68.7195	67.5707	1.14876
1.7	0.00843257	0.00858869	0.000156122	118.588	116.432	2.15565
1.8	0.00504136	0.0051435	0.000102139	198.359	194.42	3.939
1.9	0.00309897	0.00316802	0.0000690475	322.688	315.655	7.03303
2.0	0.00195313	0.00200115	0.0000480287	512.0	499.712	12.2882

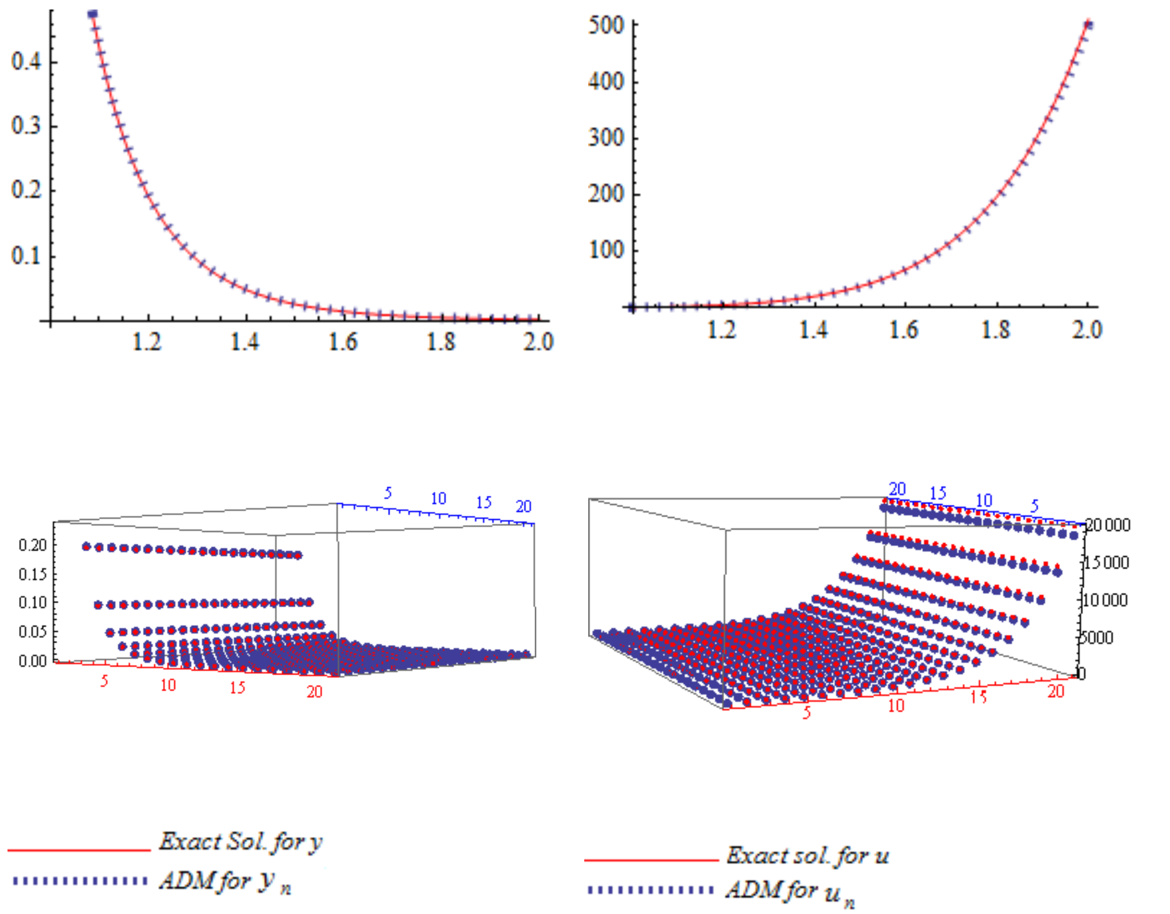

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Figure 3. comparing between approximate solution and exact solution

Example 4. Consider the following nonlinear Fractional differential equation with initial condition:

$$y''+y'-x^{\frac{9}{10}}D^{0.9}y=3!\frac{y'^2}{y}-\frac{y}{\Gamma(0.1)}+ \\ +\left(\frac{-\pi}{96}x^2+(\frac{5\sqrt{\pi}}{6}-\frac{\pi}{288})x^3+(\frac{5\sqrt{\pi}}{6}+\frac{\pi}{48\Gamma(\frac{41}{10})}-15)x^4-(3+\frac{5\sqrt{\pi}}{6\Gamma(\frac{51}{10})})x^5+\frac{360}{\Gamma(\frac{61}{10})}x^6\right)y^3 \quad (5.14)$$

subject to initial conditions: $y(1)=1.07974, y'(1)=-3.32532, y''(1)=13.6594$, with exact solution $y(x)=(x^3-\frac{\sqrt{\pi}}{24}x^2)^{-1}$, $\forall x\in[1,2]$.

To find approximate solution for nonlinear equation (5.14) by applying ADM method, first, we reduce the Bernoulli equation to the linear equation by the transformation $u=y^{-2}$ therefore the equation (5.14) will become to:

$$\frac{1}{-2}(\frac{d^2u}{dx^2}-x^{\frac{9}{10}}D^{0.9}u+\frac{du}{dx})=\frac{-\pi}{96}x^2+(\frac{5\sqrt{\pi}}{6}-\frac{\pi}{288})x^3+ \\ +(\frac{5\sqrt{\pi}}{6}+\frac{\pi}{48\Gamma(\frac{41}{10})}-15)x^4-(3+\frac{5\sqrt{\pi}}{6\Gamma(\frac{51}{10})})x^5+\frac{360}{\Gamma(\frac{61}{10})}x^6 \quad (5.15)$$

Subject to the initial conditions: $u(1)=0.85775,\ u'(1)=5.28329, u''(1)=27.1114$.
Consequently, the series solution for $u(x)$ by using the ADM described in [13] for this equation given by $u(x)=\sum_{n=0}^{\infty}u_n(x)$, therefore we have:

$$u=u(1)+(x-1)u'(1)+L^{-1}\left(x^{\frac{9}{10}}D^{0.9}u\right)-L^{-1}\left(\frac{du}{dx}\right)- \\ -2L^{-1}\left(\frac{-\pi}{96}x^2+(\frac{5\sqrt{\pi}}{6}-\frac{\pi}{288})x^3+(\frac{5\sqrt{\pi}}{6}+\frac{\pi}{48\Gamma(\frac{41}{10})}-15)x^4-(3+\frac{5\sqrt{\pi}}{6\Gamma(\frac{51}{10})})x^5+\frac{360}{\Gamma(\frac{61}{10})}x^6\right)$$

Consequently, we obtained the results approximate solutions for eq.(5.15) through the iteration formula of this equation which is given by:

$$u_{n+1}=u(1)+(x-1)u'(1)+\int_1^x\int_1^x\left(x^{\frac{9}{10}}D^{0.9}u_n(x)-\frac{du_n(x)}{dx}\right)dxdx-2\int_1^x\int_1^x\left(\frac{-\pi}{96}x^2+(\frac{5\sqrt{\pi}}{6}-\frac{\pi}{288})x^3\right)dxdx \\ -2\int_1^x\int_1^x\left((\frac{5\sqrt{\pi}}{6}+\frac{\pi}{48\Gamma(\frac{41}{10})}-15)x^4-(3+\frac{5\sqrt{\pi}}{6\Gamma(\frac{51}{10})})x^5+\frac{360}{\Gamma(\frac{61}{10})}x^6\right)dxdx$$

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Hence, Table 4 shows the approximate solution y(x) ≈ $\sum_{n=0}^{4} y_n(x)$ for Equation (5.14) and the approximate solution u(x) ≈ $\sum_{n=0}^{4} u_n(x)$ for Equation (5.15) obtained by using ADM method has been plotted in Figure 4. It is to be noted that only the three iterations were used in evaluating the approximate solution for Fig. 4.

x_i	Exact solution	ADM	Absolute value	Exact solution	ADM	Absolute value
	$y(x_i)$	y_n	$\lvert y-y_n \rvert$	$u(x_i)$	$u_n(x_i)$	$\lvert u-u_n \rvert$
1.0	1.079741	1.0797411	0.000001	0.85775	0.85775	3.3395×10^{-7}
1.1	0.8053872	0.80538723	0.00000003	1.54166687	1.54166679	8.1617×10^{-8}
1.2	0.6166548	0.61665488	0.00000008	2.62975770	2.629757	4.9716×10^{-7}
1.3	0.4825813	0.48258135	0.00000005	4.29397018	4.293969	9.2190×10^{-7}
1.4	0.3847264	0.38472646	0.00000006	6.75609849	6.756097	1.3699×10^{-6}
1.5	0.3116398	0.311639858	0.000000058	10.2966057	10.296604	1.8627×10^{-6}
1.6	0.2559549	0.255954923	0.000000023	15.2641665	15.264264	2.4393×10^{-6}
1.7	0.21278556	0.212785576	0.000000016	22.0859301	22.085927	3.18186×10^{-6}
1.8	0.17880391	0.178803927	0.000000017	31.2785028	31.278499	4.29043×10^{-6}
1.9	0.151689978	0.151689988	0.00000001	43.4596507	43.459644	6.52324×10^{-6}
2.0	0.129792743	0.129792757	0.000000014	59.3607228	59.3607093	0.0000135429

Table 4.The absolute errors of eq.(5.14) between the approximate values by ADM and exact solution

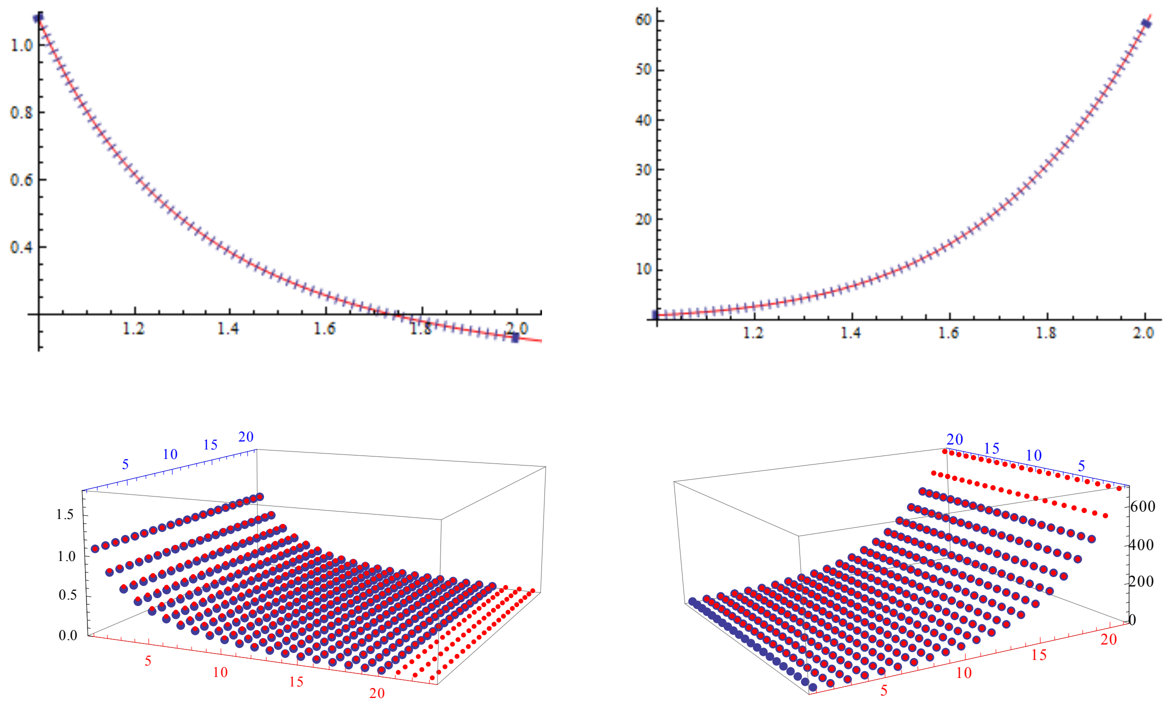

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Figure 4. comparing between approximate solution and exact solution

6. Conclusion

In this paper, the Adomian decomposition method has been successfully employed to obtain the approximate solutions of second-order the nonlinear fractional differential equation for the Bernoulli equation. Moreover, the results obtained in this research showed that the approximate solutions for this type of nonlinear fractional differential equations of the Bernoulli equation, which was obtained from this method are almost the same as the analytical solutions; in addition, the method is efficient, reliable, and computationally stable.

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