Existence and Uniqueness of the approximation Solutions To the Boundary Value Problem for Fractional Sturm-Liouville Differential Equations with the Caputo Derivative

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Received: 02/01/2023	**Accepted:28/01/2023**

وجود و وحدانية الحلول التقريبية لمشكلة القيمة الحدية للمعادلات التفاضلية

شتورم لوفـــــــــيل الكسرية مع المشتق Caputo

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الملخص:

الهدف من المقال هو دراسة وجود الحل التقريبي و وحدانيته لمشكلة القيمة الحدية لمعادلة شتورم لوفـــــــــيل الكسرية مع المشتق Caputo في فضاء بنـــاخ. حيث قمنا بإثبات بعض النظريات حول وجود الحل و تفرده لـــــــــــــ FSLP و من ثم قمنا بتوسيع نظرية النقطة الثابتة لـــــــــــ ODEs لتشمل مشكلة Fractional Sturm-Liouville ذات الشروط الحدية، و بعد ذلك تم الحصول على الحل التقريبي بواسطة الطرق التقريبية و هي طريقـــــــــتي بيكــــــارد و مان –كراسنوسلســـــــكي التكـــرارية.

ABSTRACT

In this paper, the researcher investigated the Fractional Sturm–Liouville boundary value problem with the Caputo derivative and studied the existence and uniqueness of its solution in Banach space, in addition to the continuation of its solution. As the result, researcher proved some theorems on the existence of solutions for FSLP and then extend a Fixed-Point theorem for ODEs to this of the Fractional Sturm–Liouville problem with boundary conditions. Also, the given problem by obtained via the constructing approximate solution by Picard and Krasnoselskij-Mann iterations.

Keywords: Fractional Sturm–Liouville Problem, Caputo fractional derivatives, iterative methods, contraction and non-expansive mapping, Fixed-Point theorem.

1. INTRODUCTION

We consider the Fractional Sturm–Liouville differential problem with boundary conditions as following:

$$-{}^{c}D_{b-}^{\alpha}\left(p(x)\,{}^{c}D_{a+}^{\alpha}u\right)(x)+q(x)u(x)+f(x,u(x))=0 \qquad (1.1)$$

$$\alpha_1 u(\mathrm{a}) + \alpha_2 I_{b-}^{1-\alpha}\left(p\,{}^{c}D_{a+}^{\alpha}u\right)|_{x=a}\,(x) = 0 \tag{1.2}$$
$$\beta_1 u(b) + \beta_2 I_{b-}^{1-\alpha}\left(p\,{}^{c}D_{a+}^{\alpha}u\right)|_{x=b}\,(x) = 0$$

where $\frac{1}{2} < \alpha \le 1, {}^{c}D_{a+}^{\alpha}, {}^{c}D_{b-}^{\alpha}$ are denote the Caputo fractional derivatives, $u(x) \in C(I,\Re)$, $C(I,\Re)$ set of all continuous functions from I to $\Re$ with the norm $\|u\|_{\infty} = \sup\{u(x) : x \in I\}$, consequently, $\left(C(I,\Re), \|\ \|_{\infty}\right)$ is a Banach space, $p(x) \in C^1(I,\Re)$ and $q(x) > 0$ is absolute continuous function on $I = [a,b]$ with $p(x) > 0$ for all $x \in I$, $\alpha_i, \beta_i, i = 1,2$ are real constants, $f : I \times \Re \to \Re$ is defined and differentiable on the interval I, where f satisfied Lipschitzian condition, i.e., there exist constant $L > 0$ such that $\|f(x,u) - f(x,v)\| \le L\|u - v\|$ for any $x \in I$, $u,v \in C(I,\Re)$, L is Lipschitzian constant.

The fractional calculus has allowed the operations of integration and differentiation fractional order. So, (Machado et al., 2011) introduced the history of the fractional calculus, and the theory of fraction differential equations effected many by authors in mathematics, physics and engineering, (see the papers: [11,12,13,15,17,34,35,36]). The existence and uniqueness of the solution for fractional differential equations have been studied by authors in [4,6,7,10, 14,18, 23,24,46,47]. (Abbas, 2011) discussed the existence and uniqueness of solution to fractional order ordinary and delay differential.

(Pandey et al., 2020) presented the regular Fractional Sturm–Liouville Problem of order μ ($0 < \mu < 1$) , where the authors was applying a fractional variational method to studying the Sturm–Liouville eigenvalues and eigenfunctions with the Caputo fractional derivatives.

(Klimek et al., 2016) proved the existence of strong solutions for space-time fractional diffusion equations in bounded domain by using the method of separating variables that was depending on the Fractional Sturm–Liouville theory. (El-Sayed, 2019) studied the existence and uniqueness of a solution for a Sturm–Liouville fractional differential equation with a multi-point boundary condition via the Caputo derivative; existence and uniqueness results for the given problem are obtained using Banach Fixed-point Theorem.

The problem of the existence and uniqueness of the solution for Fractional Sturm–Liouville have been considered by many authors; see results in [22]. (Klimek et al., 2018) discussed the exact and numerical solutions for the fractional Sturm–Liouville problem in a bounded domain. The derived Fractional Sturm–Liouville equations with corresponding boundary conditions contain the differential operator, which is a composition of the left and the right fractional derivative.

Many authors studied these types of the Fractional Sturm–Liouville operators. For instance, (Klimek & Agrawal, 2012) investigated the eigenvalue and eigenfunction properties of both the regular and the singular Fractional Sturm–Liouville theory; in addition, (Klimek & Agrawal, 2013) defined Fractional Sturm–Liouville operators containing left and right Sturm–Liouville, and left and right Caputo fractional derivatives.

(Rivero et al., 2013) studied some of the basic properties of the Sturm–Liouville theory for fractional operators involving Riemann-Liouville, Caputo or Liouville fractional operators. (Ciesielski et al., 2017) introduced the developed numerical method for solving a fractional eigenvalue problem the version of the Fractional Sturm–Liouville problem with the homogeneous mixed boundary conditions. (Batiha et al., 2022) Purposed investigate the existence and uniqueness of solutions for generalized Sturm–Liouville and Langevin equations

formulated using Caputo–Hadamard fractional derivative operator in accordance with three nonlocals Hadamard fractional integral boundary conditions.
On the other hand, the iteration methods of Picard, Mann and Ishikawa iterations are used to solving the problems for partial and differential equations.These iterative processes have been extensively studied and applied by many authors. Such as, (Vasile B., 2004) presented a study was that stated that the iterative process of the Picard iteration converges faster than Man iteration. (Park, 1994) studied the Mann iteration process can applied to approximate the fixed point of strictly pseudo contractive mapping in Banach spaces. (Olaleru, 2009) investigated the convergence rate of the Picard, Mann and Ishikawa iteration when the operators are generalized contractive operators. Addition there are many study on the convergence theorems and stability problems in Banach spaces and metric spaces using the Mann's iteration scheme or the Ishikawa's iteration scheme (see, [8,9,31,33,39,41]).
The rest of this article is organized as follows: In Section 2 & 3 we introduce some basic definitions and previously known results that, which will be used throughout this paper. In Section 4, we have given the main results, where we discussed the existence solution for Fractional Sturm–Liouville boundary value problem (1.1)-(1.2) and present two continuation theorems for FSLP, which are generalization of the continuation theorems for ODEs.

2. PRELIMINARIES

In this section, we recall some basic definitions, notations and some properties about fractional calculus operators, based on the following books [5,11,12]:

Definition 2.1. Let $\alpha > 0$ and function $f:\Re^+ \to \Re$. The left and right Riemann–Liouville fractional integrals operator I_{a+}^{α} and I_{b-}^{α} of order $\alpha \in \Re^+$ of f are defined by:

$$I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)ds, \quad x \in (a,b] \tag{2.1}$$

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s)ds, \quad x \in [a,b) \tag{2.2}$$

respectively, provided the integral exists, where $\Gamma(.)$ is the Euler gamma function, which is defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

Definition 2.2. The left Riemann–Liouville fractional derivative of order $\alpha \in \Re^+$ $(0<\alpha<1)$ of function f denoted by $D_{a+}^{\alpha} f$ is defined by:

$$D_{a+}^{\alpha} f(x) := D I_{a+}^{1-\alpha} f(x), \qquad \forall x \in (a,b] \tag{2.3}$$

Similarly, the right Riemann–Liouville fractional derivative of order $\alpha \in \Re^+$ $(0<\alpha<1)$ of function f denoted by $D_{b-}^{\alpha} f$ is defined by:

$$D_{b-}^{\alpha} f(x) := -D I_{b-}^{1-\alpha} f(x), \qquad \forall x \in [a,b) \tag{2.4}$$

Definition 2.3. The Caputo derivative of order α for function $f:\Re^+ \to \Re$ is given by:

$${}^{C}D_{0^+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (x-s)^{n-\alpha-1} f^{(n)}(s)ds \tag{2.5}$$

Provided the right side is positive defined on $\Re^+$ where $n \in \aleph$ with $n-1<\alpha<n$.

Remark 2.1. if $\alpha = n \in \aleph$, then Caputo derivative becomes ${}^{C}D^{\alpha} f(x) = f^{(n)}(x)$.

Remark 2.2. If $f(x) \in C^n[0,\infty]$, then

$$ {}^{c}D_{0^+}^{\alpha} f(x) = \frac{1}{\Gamma(n-1)} \int_0^x \frac{f^{(n)}(s)}{(x-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(k)}(x), \tag{2.6} $$

where $x > 0, n-1 < \alpha < n$.

Definition 2.4. The left and the right Caputo fractional derivatives of order $(0 < \alpha < 1)$ are given by:

$$ {}^{c}D_{a+}^{\alpha} f(x) := D_{a+}^{\alpha}[f(x) - f(a)], \quad \forall x \in (a,b] \tag{2.7} $$

$$ {}^{c}D_{b-}^{\alpha} f(x) := D_{b-}^{\alpha}[f(x) - f(b)], \quad \forall x \in [a,b). $$

Definition 2.5. Let $AC[a,b]$ be the space of the functions f, which are absolutely continuous on $[a,b]$. We denote $AC^n[a,b]$ by the set of the functions f, which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)} \in AC[a,b]$.

Remark 2.3. Let $AC[0,1]$ be the space of the functions f, which are absolutely continuous on $[0,1]$. We denote $AC^n[0,1]$ by the set of the functions f, which have continuous derivatives up to order $n-1$ on $[0,1]$ such that $f^{(n-1)} \in AC[0,1]$. In particular $AC^1[0,1] = AC[0,1]$

Definition 2.6. If f is absolutely continuous in interval $[a, b]$, then the above Caputo fractional derivatives satisfy, almost everywhere on $[a, b]$, the following relations:

$$ {}^{c}D_{a+}^{\alpha} f(x) := I_{a+}^{1-\alpha} f(x) \text{ and } {}^{c}D_{b-}^{\alpha} f(x) := -I_{b-}^{1-\alpha} f(x) $$

Lemma 2.1. If $f \in AC^n[0,1]$, then the Caputo fractional derivative ${}^{c}D^{\alpha} f(t)$ exists almost everywhere on $[a,b]$, where n is the smallest integer greater than or equal to α.

In the following, we recall some results for the fractional calculus operators.

Proposition 2.1. Let $\alpha, \beta > 0$ and $f \in L^p(a,b), \ (1 \le p \le \infty)$. Then the following equations:

$$ I_{a+}^{\alpha} I_{a+}^{\beta} f(x) := I_{a+}^{\alpha+\beta} f(x) \text{ and } I_{b-}^{\alpha} I_{b-}^{\beta} f(x) := I_{b-}^{\alpha+\beta} f(x) $$

are satisfied almost everywhere in $[a,b]$. If function f is continuous, then composition rules hold for all $x \in [a,b]$.

Proposition 2.2. Let $0 < \beta < \alpha$ and $f \in L^p(a,b), \ (1 \le p \le \infty)$. Then the following equations:

$$ D_{a+}^{\beta} I_{a+}^{\alpha} f(x) := I_{a+}^{\alpha-\beta} f(x) \quad \text{and } D_{b-}^{\beta} I_{b-}^{\alpha} f(x) := I_{b-}^{\alpha-\beta} f(x) $$

are satisfied for almost all $x \in [a,b]$. If function f is continuous, then composition rules hold for all $x \in [a,b]$.

Proposition 2.3. If $\alpha > 0$ and $f \in L^p(a,b), \ (1 \le p \le \infty)$. then the following is true:

$$ D_{a+}^{\alpha} I_{a+}^{\alpha} f(x) := f(x) \text{ and } D_{b-}^{\alpha} I_{b-}^{\alpha} f(x) := f(x), $$

For almost all $x \in [a,b]$. If function f is continuous, then composition rules hold for all $x \in [a,b]$.

Proposition 2.4. If f is continuous in interval $[a, b]$ and $\alpha > 0$, then:

$$ {}^{c}D_{a+}^{\alpha} I_{a+}^{\alpha} f(x) := f(x) \text{ and } {}^{c}D_{b-}^{\alpha} I_{b-}^{\alpha} f(x) = f(x) $$

Proposition 2.5. Let $0 < \alpha \le 1$. If f is absolutely continuous in interval $[a, b]$ (i.e., $f \in AC[a,b]$), then almost everywhere on $[a,b]$:

$$ I_{a+}^{\alpha}\, {}^{c}D_{a+}^{\alpha} f(x) := f(x) - f(a) \text{ and } I_{b-}^{\alpha}\, {}^{c}D_{b-}^{\alpha} f(x) = f(x) - f(b). $$

Proposition 2.6. If $f \in L^1(a,b)$ and $I_{a+}^{1-\alpha} f, I_{b-}^{1-\alpha} f \in AC[a,b]$, then the following are true:

$$I_{a+}^{\alpha}\,{}^{c}D_{a+}^{\alpha}f(x)=f(x)-\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}I_{a+}^{1-\alpha}f(a),$$

$$I_{b-}^{\alpha}\,{}^{c}D_{b-}^{\alpha}f(x)=f(x)-\frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}I_{b-}^{1-\alpha}f(b),$$

almost everywhere on $[a,b]$.

Proposition 2.7. Let $\alpha>0, p\geq 1, q\geq 1$ and $\frac{1}{p}+\frac{1}{q}\leq 1+\alpha$ ($p\neq 1$ and $q\neq 1$ in the case when $\frac{1}{p}+\frac{1}{q}\leq 1+\alpha$). If $f\in L^{p}(a,b)$ and $g\in L^{q}(a,b)$, then

$$\int_{a}^{b}f(x)I_{a+}^{\alpha}g(x)dx=\int_{a}^{b}g(x)I_{b-}^{\alpha}f(x)dx.$$

Proposition 2.8. Assume that $0<\alpha<1$, $f\in AC[a,b]$ and $g\in L^{q}(a,b)$, $1\leq p\leq\infty$, then the following integration by parts formula

$$\int_{a}^{b}f(x)D_{a+}^{\alpha}g(x)dx=\int_{a}^{b}g(x)D_{b-}^{\alpha}f(x)dx+f(x)I_{a+}^{\alpha}g(x)\Big|_{x=a}^{x=b}$$

holds.

3. FIXED POINT THEOREMS IN BANACH SPACE

Definition 3.1. [44,45] Let E be a real Banach space, K a nonempty convex subset of E. Let $T:K\to K$ be a mapping. Given an $x_0\in K$ and a real number $\lambda\in[0,1]$, the sequence $\{x_n\}\subset K$ defined by the formula:

$$x_{n+1}=Tx_n,\qquad n=0,1,2,... \tag{3.1}$$

is called Picard's iterationin1890 [16], and the sequence $\{x_n\}$ defined by the formula:

$$x_{n+1}=(1-\lambda)x_n+\lambda Tx_n,\qquad n=0,1,2,... \tag{3.2}$$

is called the Krasnoselskij iteration, or Krasnoselskij–Mann's iteration is defined by[42].

Clearly, the Mann iteration (3.2) reduces to sequence $x_{n+1}=\frac{1}{2}(T(x_n)+x_n)$, when $\lambda=\frac{1}{2}$, and (3.2) reduces to the Picard iteration for $\lambda=1$.

For $y_0\in K$, the sequence $\{y_n\}\subset K$ defined by the following formula:

$$y_{n+1}=(1-\lambda_n)y_n+\lambda_n Ty_n,\ n=0,1,... \tag{3.3}$$

called the Mann's iteration, where $\lambda_n\subset[0,1]$ is a sequence of real numbers satisfying the following conditions:

1. $\lambda_0=1$
2. $0\leq\lambda_n<1, \forall n\in\aleph$
3. $\sum_{n}\lambda_n=\infty$

Definition 3.2 [44,45,8] Let K a nonempty convex subset of Banach space E. Then a mapping $T:K\to K$ is said to:

(i) Non-expansive mapping if

$$\|Tx-Ty\|\leq\|x-y\|\forall x,y\in K$$

(ii) Contraction mapping if

$$\|Tx-Ty\|\leq L\|x-y\|\forall x,y\in K$$

where the constant L is recall as Lipschitz constant of T.

Theorem 3.1. If K is a nonempty closed convex and bounded subset of a uniformly convex Banach space E then any non-expansive mapping $T:K \to K$ has a fixed point.

Definition 3.3. Let a_n and b_n be two sequences of positive numbers that converge to a,b respectively. Assume that there exists the following limit

$$\lim_{n\to\infty}\frac{|a_n+a|}{|b_n+b|}=l$$

(i) If $l=0$, then it said that $\{a_n\}$ converge faster to a than $\{b_n\}$ to b.

(ii) If $0<l<\infty$, then it said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Definition 3.4. Suppose that we have two iteration sequences $\{x_n\}$ and $\{y_n\}$ both converging to a fixed point p. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers, such that:

$$d(x_n,p)\le a_n \text{ for all } n\in\aleph,$$
$$d(y_n,p)\le b_n \text{ for all } n\in\aleph,$$

where $\{a_n\}$ and $\{b_n\}$ converging to 0. If $\{a_n\}$ converge faster than $\{b_n\}$ in the sense of (Def.3.3), then $\{x_n\}$ is said to converge faster than $\{y_n\}$ to p.

Definition 3.5. If $\{x_n\}$ and $\{y_n\}$ are two iterative sequences that converge to the unique fixed point p of T, then $\{x_n\}$ converges faster than $\{y_n\}$, if

$$\lim_{n\to\infty}\frac{d(x_n,p)}{d(y_n,p)}=0.$$

Remark 3.1. For each $x,y,z\in E$ and $\lambda\in[0,1]$, we have that:

$$d(z,W(x,y,\lambda))\ge(1-\lambda)d(z,y)-\lambda d(z,x).$$

Consequently, we recall the basic fixed point iteration which appears in Banach contraction principle, that is Picard iteration: $x_{n+1}=Tx_n$ for all $n\in\aleph$, furthermore, for each $n\in\aleph$; we get the implicit Mann iteration: $x_{n+1}=W(Tx_n,x_n,\alpha_n)$.

Theorem3.2.[42,43,9] Let K a subset of Banach space E and $T:K\to K$ be a nonexpansive mapping. For an arbitrary $y_0\in K$, consider the Mann iteration process $\{y_n\}$ given by (3.3) under the following assumptions:

(a) $y_n\in K$ for positive integers n;

(b) $0\le\lambda_n<b<1$ for positive integers n;

(c) $\sum_n\lambda_n=+\infty$

If $\{y_n\}$ is bounded, then $y_n-Ty_n\to 0$ as $n\to\infty$.

Theorem 3.3.[48] Let K a compact convex subset of a real Banach space E, and T be a nonexpansive mapping on K. Let $y_0\in K$ and define a sequence $\{y_n\}$ in K by

$$y_{n+1}=(1-\lambda_n)y_n+\lambda_n Ty_n, \qquad n=0,1,2,...$$

where λ_n is a sequence in the interval $[0,1]$, such that $\sum_n^{\infty}\lambda_n=\infty$ and $\limsup_n\lambda_n<1$. Then $\{y_n\}$ converges strongly to the fixed point p of T.

We present the following corollaries of the Theorem 3.2.

Corollary 3.1. Let K be a convex and compact subset of a Banach space E and $T:K\to K$ be a non-expansive mapping. If the Mann iteration process $\{y_n\}$ given by(3.3) satisfies assumptions (a)-(c) of Theorem 3.2, then $\{y_n\}$ converges strongly to a fixed point of T .

Corollary 3.2. Let E be a real normed space, K a closed bounded convex subset of E and let $T:K\to K$ be a non-expansive mapping. If $I-T$ maps closed bounded subset of E into closed subset of E and $\{x_n\}$ is the Mann iteration defined by(3.3) with $\{\lambda_n\}$ satisfies assumptions(a)-(c) of Theorem 3.2, then $\{x_n\}$ converges strongly to a fixed point of T in K.

Theorem 3.4. [3] **(*Banach's Fixed Point Theorem*).**
Let K be a non-empty closed subset of a Banach space E , then any contraction mapping T of K into itself has a unique fixed point, i.e. there exists a unique $x\in K$ such that $Tx=x$.
Theorem 3.5. [3] **(Schaefer's *Fixed Point Theorem)*.**
Let E be a Banach space, and F of E into itself a completely continuous operator. If the set:

$$\varepsilon=\{y\in F: y=\lambda Fy\ ,\ for\,some\ \ \lambda\in(0,1)\}$$

Is bounded, then F has fixed point.
Let E be a Banach space and K a subset of E . An operator $T:K\to E$ is called compact if it is continuous and maps bounded subsets to relative compact sets. Below is the Schauder Fixed point theorem.

Theorem 3.6. [32] **(Schauder *Fixed Point Theorem*)**
Let K be a closed bounded convex subset of a Banach space E. Assume that $T:K\to K$ is compact. Then T has at least one fixed point in K.

4. MAIN RESULT

We discuss the existence and approximate of solutions of Fractional Sturm–Liouville differential Problem (1.1) subject to boundary conditions (1.2) in the following lemma:

Lemma 4.1. Let $I=[a,b]$, $\frac{1}{2}<\alpha\le 1$ and let $p:I\to\Re, q:I\to\Re, r:I\to\Re$ are continuous functions, such that $p(x)>0$, $r(x)>0$ for all $x\in I$ and $\alpha_i,\beta_i, i=1,2$ are constants. A function u is a solution of the Fractional integral equation:

$$u(x)=u(a)+\frac{\beta_1 u(b)(b-x)^{\alpha}}{\beta_2\Gamma(\alpha+1)}I^{\alpha}_{a+}p^{-1}(x)+I^{\alpha}_{a+}\left(\frac{1}{p(t)}I^{\alpha}_{b-}\big(q(t)u(t)+f(t,u(t))\big)\right) \qquad (4.1)$$

if and only if u is a solution of Fractional Sturm–Leoville boundary value problem (1.1)-(1.2).
Proof. Assume u satisfied (1.1) and (1.2), then by operating by I^{α}_{b+} on both side equation (4.1), we obtain:

$$-\left(p(x)\,{}^{c}D^{\alpha}_{a+}u\right)(x)+I^{\alpha}_{b+}q(x)u(x)+I^{\alpha}_{b+}f(x,u(x))=c \qquad (4.2)$$

Consequently;

$$-\left(p(x)\,{}^{c}D^{\alpha}_{a+}u\right)(x)+I^{\alpha}_{b+}\big(q(s)u(s)+f(s,u(s))\big)=c$$

Furthermore, since $p(x)>0$ then:

$${}^{c}D^{\alpha}_{a+}u(x)=\frac{1}{p(x)\Gamma(\alpha)}\int_x^b(s-x)^{\alpha-1}\big(q(s)u(s)+f(s,u(s))\big)ds-\frac{c}{p(x)} \qquad (4.3)$$

when $x=b$ we get ${}^{c}D_{a+}^{\alpha}u(b)=\dfrac{c^{*}}{p(b)}\Rightarrow c^{*}=p(b){}^{c}D_{a+}^{\alpha}u(b),\ c^{*}=-c$, subsequently, we get the following :

$$ {}^{c}D_{a+}^{\alpha}u(x)=\frac{1}{p(x)\Gamma(\alpha)}\int_{x}^{b}(s-x)^{\alpha-1}\left(q(s)u(s)+f(s,u(s))\right)ds+\frac{p(b){}^{c}D_{a+}^{\alpha}u(b)}{p(x)} \tag{4.4} $$

$$ I_{a+}^{\alpha}\,{}^{c}D_{a+}^{\alpha}u(x)=I_{a+}^{\alpha}\left(\frac{1}{p(x)\Gamma(\alpha)}\int_{x}^{b}(s-x)^{\alpha-1}\left(q(s)u(s)+f(s,u(s))\right)ds+\frac{p(b){}^{c}D_{a+}^{\alpha}u(b)}{p(x)}\right) $$

$$ u(x)|_{x=a}=I_{a+}^{\alpha}\left(\frac{p(a){}^{c}D_{a+}^{\alpha}u(b)}{p(x)}\right)+I_{a+}^{\alpha}\left(\frac{1}{p(x)\Gamma(\alpha)}\int_{x}^{b}(s-x)^{\alpha-1}\left(q(s)u(s)+f(s,u(s))\right)ds\right) $$

$$ u(x)=u(a)+p(a){}^{c}D_{a+}^{\alpha}u(b)I_{a+}^{\alpha}\left(\frac{1}{p(x)}\right)+I_{a+}^{\alpha}\left(\frac{1}{p(x)\Gamma(\alpha)}\int_{x}^{b}(s-x)^{\alpha-1}\left(q(s)u(s)+f(s,u(s))\right)ds\right) $$

where: ${}^{c}D_{a+}^{\alpha}u(b)=\dfrac{\beta_1(b-x)^{\alpha}}{\beta_2 p(b)\alpha\Gamma(\alpha)}$; so:

$$ u(x)=u(a)+\frac{\beta_1(b-x)^{\alpha}u(b)}{\beta_2\alpha\,\Gamma(\alpha)}\left(\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{(x-t)^{\alpha-1}}{p(t)}dt\right) $$
$$ +\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left(\frac{1}{p(t)\Gamma(\alpha)}\int_{t}^{b}(s-t)^{\alpha-1}[q(s)u(s)+f(s,u(s))]ds\right)dt $$

we can rewrite the previous formula as the form:

$$ u(x)=u(a)+\frac{\beta_1(b-x)^{\alpha}u(b)}{\beta_2\alpha\Gamma(\alpha)}I_{a+}^{\alpha}p^{-1}(x)+I_{a+}^{\alpha}\left(\frac{1}{p(t)}I_{b-}^{\alpha}\left(q(t)u(t)+f(t,u(t))\right)\right) \tag{4.5} $$

Theorem 4.1. Assume that the following conditions are satisfied:

(***H₁***) the function $f:I\times\Re\rightarrow\Re$ is continuous.

(***H₂***) There exist constants $L>0$ and $0<L<1$ such that $\left|f(x,u_1(x))-f(x,u_2(x))\right|\le L\left|u_1-u_2\right|$ for any $u_1,u_2\in C(I,\Re)$ and $x\in I$. There exist positive constant Q such that $\left|q(t)\right|<Q$ for all $x\in I$. If

$$ \sigma=(Q+L)I_{a+}^{\alpha}\left(\frac{1}{p(t)}\frac{(b-t)^{\alpha}}{\Gamma(\alpha+1)}\right)<1 \tag{4.6} $$

Then there exists a unique solution for Fractional Sturm–Louiville boundary value problem on I.

Proof. Transform problem (1.1)-(1.2) into a fixed point problem, thus, consider the operator $T:C(I,\Re)\rightarrow C(I,\Re)$ defined by:

$$ Tu(x)=u(a)+\frac{\beta_1(b-x)^{\alpha}u(b)}{\beta_2\alpha\Gamma(\alpha)}\left(\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{(x-t)^{\alpha-1}}{p(t)}dt\right) $$
$$ +\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left(\frac{1}{p(t)\Gamma(\alpha)}\int_{t}^{b}(s-t)^{\alpha-1}[q(s)u(s)+f(s,u(s))]ds\right)dt \tag{4.7} $$

Journal of Humanitarian and Applied Sciences - مجلة العلوم الإنسانية والتطبيقية
(رقم الإيداع المحلي 95 / 2020) - (ISSN: 2706- 9087) - (معامل التأثير العربي شهادة رقم: 472 - 2020)
كلية الآداب والعلوم قصر الأخيار - جـامعة المرقب
Volume 8 – Issue 15 المجلد 8 العدد 15

Obviously, any fixed point of operator T is solution for the problem (1.1)–(1.3).

To prove that the T operator has a fixed point, we should use the Banach contraction principle theorem. So, let $u_1, u_2 \in C(I,\Re)$. Then for $x \in I$,we have

$$|Tu_1(x)-Tu_2(x)| \le I_{a+}^{\alpha}\left(\frac{1}{p(t)}\left(I_{b-}^{\alpha}\left(|q(t)||u_1(t)-u_2(t)|+|f(s,u_1(t))-f(s,u_2(t))|\right)\right)\right) \tag{4.8}$$

Consequently,

$$\begin{aligned}|Tu_1(x)-Tu_2(x)| &\le I_{a+}^{\alpha}\left(\frac{1}{p(t)}\left(I_{b-}^{\alpha}\left(|q(t)||u_1(t)-u_2(t)|+L|u_1(t)-u_2(t)|\right)\right)\right)\\ &\le I_{a+}^{\alpha}\left(\frac{1}{p(t)}\left(I_{b-}^{\alpha}\left((Q+L)|u_1(t)-u_2(t)|\right)\right)\right)\\ &\le (Q+L)I_{a+}^{\alpha}\left(\frac{1}{p(t)}\frac{(b-t)^{\alpha}}{\Gamma(\alpha+1)}\right)\|u_1-u_2\|_{\infty}\end{aligned} \tag{4.9}$$

Consequently:

$$|Tu_1(x)-Tu_2(x)| \le \sigma\|u_1-u_2\|_{\infty}$$

Since $0<\sigma<1$ and so by (4.6), we obtain T is a contraction mapping on I. As a consequence of Banach's fixed-point Theorem 3.4 for operators deducible that the operator T has a unique fixed point on I, which implies that the fractional Sturm–Louiville problem has a unique solution on I. This completes the proof.

Theorem 3.2. Assume that a function $f: I\times\Re\to\Re$ is continuous, and there exist a constant $M>0$ such that $\|f(x,u)\|\le M$ for any $u\in C(I,\Re)$ and $x\in I$. There exist constants $Q>0$, $N>0$ such that $|q(t)|<Q$ for all $t\in I$, and if

$$I_{a+}^{\alpha}\left(\frac{(b-t)^{\alpha}}{p(t)}\right)\le N \tag{4.10}$$

Then the Fractional Sturm–Louivilli differential equation with the boundary conditions has at least one unique solution on I.

Proof. We shall use the Schaefer's fixed point Theorem 3.5 to prove that T defined by (4.7) has a fixed point.

Firstly: we show that T is a continuous. Let $\{u_n\}$ be a sequence such $u_n\to u$ in $C(I,\Re)$. Then for each $x\in I$ we have:

$$\begin{aligned}|Tu_n(x)-Tu(x)| &\le I_{a+}^{\alpha}\left(\frac{1}{p(t)}\left(I_{b-}^{\alpha}\left(|q(t)||u_n(t)-u(t)|+|f(s,u_n(t))-f(s,u(t))|\right)\right)\right)\\ &\le I_{a+}^{\alpha}\left(\frac{1}{p(t)}\left(I_{b-}^{\alpha}\left(|q(t)||u_n(t)-u(t)|+\sup_{t\in I}|f(s,u_n(t))-f(s,u(t))|\right)\right)\right)\\ &\le I_{a+}^{\alpha}\left(\frac{1}{p(t)}\left(I_{b-}^{\alpha}\left(\left(Q|u_n(t)-u(t)|+\sup_{t\in I}|f(s,u_n(t))-f(s,u(t))|\right)\right)\right)\right)\end{aligned}$$

$$|Tu_n(x)-Tu(x)| \le I_{a+}^{\alpha}\left(\frac{1}{p(t)}\left(I_{b-}^{\alpha}\left(\left(Q\|u_1-u_2\|_\infty+\|f(s,u_n(t))-f(s,u(t))\|_\infty\right)\right)\right)\right) \tag{4.11}$$

$$\le \frac{1}{\Gamma(\alpha+1)}\left(Q\|u_n-u\|_\infty+\|f(s,u_n(t))-f(s,u(t))\|_\infty\right)I_{a+}^{\alpha}\left(\frac{(b-t)^\alpha}{p(t)}\right)$$

Consequently, by (4.10), we get:

$$|Tu_n(x)-Tu(x)| \le \frac{1}{\Gamma(\alpha+1)}\left(Q\|u_n-u\|_\infty+\|f(s,u_n(t))-f(s,u(t))\|_\infty\right)N \tag{4.12}$$

Since f is continuous function and $u \in C(I,\Re), u_n \to u$ as $n \to \infty$, and $\|f(x,u)\| \le M$ for each $x \in I$, so we have: $\|f(s,u_n(t))\| \le M$ and $\|f(s,u(t))\| \le M$, then we have

$$\|Tu_n(x)-Tu(x)\|_\infty \le \frac{1}{\Gamma(\alpha+1)}\left(Q\|u_n-u\|_\infty+\|f(s,u_n(t))-f(s,u(t))\|_\infty\right)N$$

$$\le \frac{1}{\Gamma(\alpha+1)}\left(Q\|u_n-u\|_\infty+\|f(s,u_n(t))\|_\infty+\|f(s,u(t))\|_\infty\right)N \tag{4.13}$$

$$\le \frac{1}{\Gamma(\alpha+1)}\left(Q\|u_n-u\|_\infty+2M\right)N$$

so:

$$\|Tu_n(x)-Tu(x)\|_\infty \le \frac{1}{\Gamma(\alpha+1)}\left(Q\|u_n-u\|_\infty+2M\right)N \to 0 \ as \ n \to \infty \tag{4.14}$$

Therefore, $Tu \in C(I,\Re)$ for any $u \in C(I,\Re)$, hence T is continuous.

***Secondly*: T maps bounded sets into bounded sets in $C(I,\Re)$, it's sufficient to show that for any $\varepsilon > 0$ there exists a positive constant l such that for each $u \in \Omega_\varepsilon$ we have $\|T(u)\|_\infty < l$; where $\Omega_\varepsilon = \{u \in C(I,\Re) : \|u\|_\infty < \varepsilon\}$. Since f is a continuous function, thus for each $x \in I$ we have:

$$|Tu(x)| \le |u(a)| + \frac{\beta_1|u(b)|}{\beta_2\Gamma(\alpha+1)}I_{a+}^{\alpha}\left(\frac{|(b-x)^\alpha|}{p(x)}\right)+I_{a+}^{\alpha}\left(\frac{1}{p(t)}I_{b-}^{\alpha}\left(|q(t)||u(t)|+|f(t,u(t))|\right)\right)$$

$$\le |u(a)| + \frac{\beta_1|u(b)|}{\beta_2\Gamma(\alpha+1)}I_{a+}^{\alpha}\left(\frac{|(b-x)^\alpha|}{p(x)}\right)+I_{a+}^{\alpha}\left(\frac{1}{p(t)}I_{b-}^{\alpha}\left(Q\varepsilon+M\right)\right) \tag{4.15}$$

$$\le |u(a)| + \frac{\beta_1|u(b)|}{\beta_2\Gamma(\alpha+1)}I_{a+}^{\alpha}\left(\frac{|(b-x)^\alpha|}{p(x)}\right)+\left(Q\varepsilon+M\right)I_{a+}^{\alpha}\left(\frac{|(b-t)^\alpha|}{\Gamma(\alpha+1)p(t)}\right)$$

$$\le |u(a)| + \frac{\beta_1|u(b)|}{\beta_2\Gamma(\alpha+1)}N+\frac{(Q\varepsilon+M)}{\Gamma(\alpha+1)}N$$

Consequently

$$\|Tu(t)\|_\infty \le u(a) + \frac{\beta_1 u(b)}{\beta_2\alpha\Gamma(\alpha)}N+\frac{(Q\varepsilon+M)}{\Gamma(\alpha+1)}N := \ell_1$$

Accordingly, T is a bounded.

Thirdly: T maps bounded sets into equicontinuous sets of $C(I,\Re)$. Let $x_1, x_2 \in I, x_1 < x_2$, according to previous step $\Omega_\varepsilon = \{u \in C(I,\Re) : \|u\|_\infty < \varepsilon\}$ bounded subsets of $C(I,\Re)$, let $u \in \Omega_\varepsilon$, then:

$$|Tu(x_1) - Tu(x_2)| = \left| \frac{\beta_1 u(b)(b-x)^\alpha}{\beta_2 \Gamma(\alpha+1)} \left(\frac{1}{\Gamma(\alpha)} \int_a^{x_1} \frac{(x-t)^{\alpha-1}}{p(t)} dt \right) \right.$$
$$+ \frac{1}{\Gamma(\alpha)} \int_a^{x_1} (x-t)^{\alpha-1} \left(\frac{1}{p(t)\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} \big(q(s)u(s) + f(s,u(s))\big) ds \right) dt$$
$$- \frac{\beta_1 u(b)(b-x)^\alpha}{\beta_2 \Gamma(\alpha+1)} \left(\frac{1}{\Gamma(\alpha)} \int_a^{x_2} \frac{(x-t)^{\alpha-1}}{p(t)} dt \right) \tag{4.16}$$
$$\left. - \frac{1}{\Gamma(\alpha)} \int_a^{x_2} (x-t)^{\alpha-1} \left(\frac{1}{p(t)\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} \big(q(s)u(s) + f(s,u(s))\big) ds \right) dt \right|$$

$$|Tu(x_1) - Tu(x_2)| \le \frac{\beta_1 |u(b)| |(b-x)^\alpha|}{\beta_2 \Gamma(\alpha+1)} \left(\frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} \frac{(x-t)^{\alpha-1}}{p(t)} dt \right)$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x-t)^{\alpha-1} \left(\frac{1}{p(t)\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} \big(|q(s)| \|u(s)\|_\infty + |f(s,u(s))| \big) ds \right) dt \tag{4.17}$$

$$|Tu(x_1) - Tu(x_2)| \le \frac{\beta_1 |u(b)| |(b-x)^\alpha|}{\beta_2 \Gamma(\alpha+1)} \left(\frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} \frac{(x-t)^{\alpha-1}}{p(t)} dt \right) + \frac{(Q\varepsilon + M)}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x-t)^{\alpha-1} \left(\frac{(b-t)^\alpha}{\Gamma(\alpha+1)p(t)} \right) dt \tag{4.18}$$

Since $x_1 \to x_2$, the right hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzelá-Ascoli Theorem "which says a bounded and equicontinuous sequence of functions on a compact has a uniformly convergent subsequence", then we can conclude that T from $C(I,\Re)$ into itself is completely continuous.

Fourthly: A priori bounds. Now it remains to show that the set: $\Omega_{\varepsilon,\lambda} = \{u \in \Omega_\varepsilon : u = \lambda T(u) \text{ for some } 0 < \lambda < 1\}$, is bonded. $u \in E$ then $u = \lambda T(u)$ for some $0 < \lambda < 1$. Thus, for each $x \in I$ we have:

$$u(x) = \lambda u(a) + \frac{\lambda \beta_1 u(b)(b-x)^\alpha}{\beta_2 \Gamma(\alpha+1)} \left(\frac{1}{\Gamma(\alpha)} \int_a^x \frac{(x-t)^{\alpha-1}}{p(t)} dt \right)$$
$$+ \frac{\lambda}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(\frac{1}{p(t)\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} [q(s)u(s) + f(s,u(s))] ds \right) dt \tag{4.19}$$

consequently:

$$|u(x)| = \left| \lambda u(a) + \frac{\lambda \beta_1 u(b)(b-x)^\alpha}{\beta_2 \Gamma(\alpha+1)} \left(\frac{1}{\Gamma(\alpha)} \int_a^x \frac{(x-t)^{\alpha-1}}{p(t)} dt \right) \right.$$
$$\left. + \frac{\lambda}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(\frac{1}{p(t)\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} [q(s)u(s) + f(s,u(s))] ds \right) dt \right| \tag{4.20}$$

Since $0 < \lambda < 1$, and from previous steps we get:

$$|Tu(x)| \le |u(a)| + \frac{\beta_1 |u(b)|}{\beta_2 \Gamma(\alpha+1)} I_{a+}^{\alpha}\left(\frac{|(b-x)^{\alpha}|}{p(x)}\right) + I_{a+}^{\alpha}\left(\frac{1}{p(t)} I_{b-}^{\alpha}\left(|q(t)||u(t)| + |f(t,u(t))|\right)\right)$$

$$\le |u(a)| + \frac{\beta_1 |u(b)|}{\beta_2 \Gamma(\alpha+1)} N + \frac{(Q\varepsilon + M)}{\Gamma(\alpha+1)} N \tag{4.21}$$

Thus:

$$\|Tu(t)\|_{\infty} \le u(a) + \frac{\beta_1 u(b)}{\beta_2 \Gamma(\alpha+1)} N + \frac{(Q\varepsilon + M)}{\Gamma(\alpha+1)} N := \ell_2$$

This shows that the set $\Omega_{\varepsilon,\lambda}$ is a bounded. As consequence of Schaefer's fixed point theorem, we deduce that T has a fixed point which is a solution of the Fractional Sturm–Louiville boundary value problem (1.1) - (1.2). This completes the proof.

Theorem 4.3. Assume that all the assumptions of Theorem 4.1,Theorem 4.2 are satisfied then the unique solution u of the Fractional Sturm–Louiville boundary value problem (1.1)-(1.2) can be approximated by means of the Picard iteration u_n defined by $u_1 \in \Omega_{\varepsilon}$ arbitrary and

$$u_{n+1}(x) = u(a) + \frac{\beta_1 (b-x)^{\alpha} u(b)}{\beta_2 \alpha \Gamma(\alpha)}\left(\frac{1}{\Gamma(\alpha)}\int_a^x \frac{(x-t)^{\alpha-1}}{p(t)} dt\right) \tag{4.22}$$

$$+ \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left(\frac{1}{p(t)\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}[q(s)u_n(s) + f(s,u_n(s))]ds\right)dt \qquad \forall x \in I, n = 0,1,...$$

Theorem 4.4. Assume that the following conditions are satisfied :

1. The function $f : I \times \Re \to \Re$ is continuous.
2. There exist constant $L > 0$ such that $|f(x,u_1(x)) - f(x,u_2(x))| \le L|u_1 - u_2|$ for any $x \in I$ and $u_1, u_2 \in C(I,\Re)$.
3. There exist positive constants $Q > 0$ such that $|q(x)| < Q$ for all $x \in I$, and if

$$\sigma = (Q+L) I_{a+}^{\alpha}\left(\frac{1}{p(t)} \frac{(b-t)^{\alpha}}{\Gamma(\alpha+1)}\right) \le 1 \tag{4.23}$$

then the Fractional Sturm–Liouiville Boundary value problem (1.1)-(1.2)has at least one solution u in $\Omega_{\varepsilon} = \{u \in C(I,\Re) : \|u\|_{\infty} < \varepsilon\}$, which can be approximated by the Krasnoselskij-Mann iteration:

$$u_{n+1}(x) = (1-\mu)u_n(x) + \mu u(a) + \frac{\mu \beta_1 (b-x)^{\alpha} u(b)}{\beta_2 \alpha\ \Gamma(\alpha)}\left(\frac{1}{\Gamma(\alpha)}\int_a^x \frac{(x-t)^{\alpha-1}}{p(t)} dt\right) \tag{4.24}$$

$$+ \frac{\mu}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left(\frac{1}{p(t)\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}[q(s)u_n(s) + f(s,u_n(s))]ds\right)dt \qquad n = 0,1,...$$

where $x \in I$, $\mu \in (0,1)$ and $u_1 \in \Omega_{\varepsilon}$ is arbitrary. This completes the proof.

Proof. If $(Q+L) I_{a+}^{\alpha}\left(\frac{1}{p(t)} \frac{(b-t)^{\alpha}}{\Gamma(\alpha+1)}\right) < 1$, then the conclusion follow similarly to [Theorem8, in 2]. Therefore, we limit ourselves to the case where $(Q+L) I_{a+}^{\alpha}\left(\frac{1}{p(t)} \frac{(b-t)^{\alpha}}{\Gamma(\alpha+1)}\right) = 1$.

It follows that from [Lemma1, in 2]that Ω_ε is a non-empty convex and compact subset of the Banach space $\left(C(I,\Re),\|.\|_\infty\right)$, where $\|.\|_\infty$ is the usual supremum norm. Consider the integral operator $T:\Omega_\varepsilon \to C(I,\Re)$

$$Tu(x)=u(a)+\frac{\beta_1 u(b)}{\beta_2\Gamma(\alpha+1)}I_{a+}^{\alpha}\left(\frac{(b-x)^{\alpha}}{p(x)}\right)+I_{a+}^{\alpha}\left(\frac{1}{p(t)}I_{b-}^{\alpha}\left(q(t)u(t)+f(t,u(t))\right)\right),\quad x\in I \tag{4.25}$$

it's clear that $u\in\Omega_\varepsilon$ is solution of boundary value problem for the Fractional Sturm-Liouiville problem (1.1)-(1.2) if and only if u is a fixed pint of T ,i.e., $u=Tu$.

We first prove that Ω_ε is an invariant set with respect to T ,hence we have $T(\Omega_\varepsilon)\subset\Omega_\varepsilon$. Consequently, from pervious Theorems (4.1 &4.2), we can conclude that, for any $u\in\Omega_\varepsilon$,one has $Tu(x)\in\Omega_\varepsilon, x\in I$.

Now, for any $x_1,x_2\in I, x_1<x_2$, we have

$$|Tu(x_1)-Tu(x_2)|\le\frac{\beta_1|u(b)||(b-x)^{\alpha}|}{\beta_2\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2}\frac{(x-t)^{\alpha-1}}{p(t)}dt\right)+\frac{(Q\varepsilon+M)}{\Gamma(\alpha)}\int_{x_1}^{x_2}(x-t)^{\alpha-1}\left(\frac{(b-t)^{\alpha}}{\Gamma(\alpha+1)p(t)}\right)dt \tag{4.26}$$

Thus, $Tu\in\Omega_\varepsilon$ for all $u\in\Omega_\varepsilon$.Therefore, In addition, we conclude that T is self-mapping of Ω_ε, i.e., $T:\Omega_\varepsilon\to\Omega_\varepsilon$ and is completely continuous.

Let $u,v\in\Omega_\varepsilon$. Then for $x\in I$, then we have

$$\begin{aligned}|Tu(x)-Tv(x)|&\le I_{a+}^{\alpha}\left(\frac{1}{p(t)}\left(I_{b-}^{\alpha}\left(|q(t)||u(t)-v(t)|+L|u(t)-v(t)|\right)\right)\right)\\&\le I_{a+}^{\alpha}\left(\frac{1}{p(t)}\left(I_{b-}^{\alpha}\left((Q+L)|u(t)-v(t)|\right)\right)\right)\\&\le(Q+L)I_{a+}^{\alpha}\left(\frac{1}{p(t)}\frac{(b-t)^{\alpha}}{\Gamma(\alpha+1)}\right)\|u-v\|_\infty\end{aligned} \tag{4.27}$$

Consequently:

$$|Tu_1(x)-Tu_2(x)|\le\sigma\|u_1-u_2\|_\infty$$

According to condition (4.23), proves that T is non-expansive mapping. As a consequence of Schauder fixed-point to obtain that the operator T has a unique fixed point on I, which implies that the fractional Sturm-Louiville boundary value problem has a unique solution on I and by applying Corollary 3.1 or 3.2 we get $\{u_n\}$ converges strongly to a fixed point of T in Ω_ε. This completes the proof.

ACKNOWLEDGMENTS

The author would like to thank the anonymous referee for his/her valuable comments and suggestions, and for his/her kind support which help me to improve the manuscript.

REFRENCES

1. El-Sayed, Ahmed, and Fatma M. Gaafar. "Existence and uniqueness of solution for Sturm–Liouville fractional differential equation with multi-point boundary condition via Caputo derivative." *Advances in Difference Equations* 2019.1 (2019): 1-17. https://doi.org/10.1186/s13662-019-1976-9

2. Buica, Adriana . "Existence and continuous dependence of solutions of some functional-differential equations." *Seminar on Fixed Point Theory* 3.1 (1995): 1–14.
3. Granas, Andrzej, and James Dugundji. "Fixed point theory." Vol. 14. New York: Springer (2003).
4. Agarwal, Ravi P., Yong Zhou, and Yunyun He. "Existence of fractional neutral functional differential equations." *Compu. & Math. with Applications* 59.3(2010): 1095-1100.
5. Kilbas, Anatoliĭ Aleksandrovich, Hari M. Srivastava, and Juan J. Trujillo. "*Theory and applications of fractional differential equations*." Vol. 204. Elsevier (2006).
6. Băleanu, Dumitru, and Octavian G. Mustafa. "On the global existence of solutions to a class of fractional differential equations." *Computers & Mathematics with Applications* 59.5 (2010): 1835-1841.
7. Bayour, Benaoumeur, and Delfim FM Torres. "Existence of solution to a local fractional nonlinear differential equation." *Journal of Computational and Applied Mathematics* 312 (2017): 127-133. doi:10.1016/j.cam.2016.01.014
8. Chidume, Chidume. "Geometric properties of Banach spaces and nonlinear iterations." London: Springer (2009). doi:10.1007/978-1-84882-190-3
9. Alecsa, Cristian. "On new faster fixed point iterative schemes for contraction operators and comparison of their rate of convergence in convex metric spaces." *International Journal of Nonlinear Analysis & Applications* 8.1 (2017): 353-388 https://doi.org/10.22075/ijnaa.2017.11144.1543
10. Delbosco, Domenico, and Luigi Rodino. "Existence and uniqueness for a nonlinear fractional differential equation." *Journal of Mathematical Analysis & Applications* 204.2 (1996): 609-625. https://doi:10.1006/jmaa.1996.0456
11. Diethelm, Kai, and Neville J. Ford. "Analysis of fractional differential equations." *Journal of Mathematical Analysis & Applications* 265.2 (2002): 229-248. https://doi:10.1006/jmaa.2000.7194
12. Diethelm, Kai. "The analysis of fractional differential equations." Lect. Notes in Math. . Springer (2010). https://doi:10.1007/978-3-642-14574-2
13. Das, Shantanue, "Functional Fractional Calculus for System Identification and Controls". Springer (2008). https://doi:10.1007/978-3-540-72703-3
14. Deng, Jiqin, and Lifeng Ma. "Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations". *Applied Mathematics Letters* 23.6 (2010): 676-680. https://doi.org/10.1016/j.aml.2010.02.007
15. El-Sayed, Ahmed MA. "Fractional order differential equation*". Kyungpook Mathematical Journal*, 28(1988): 119-122.
16. Picard, Emile. "Mémoiresur la théorie des équations aux dérivéespartielles et la méthode des approximations successive". *Journal de Mathématiques pures et appliquées* 6 (1890): 145-210.
17. Hilfer, Rudolf, ed. "Applications of Fractional calculus in Physics". *World scientific*, (2000). https://doi.org/10.1142/3779
18. Ibrahim, Rabha W., and Shaher Momani "On the existence and uniqueness of solutions of a class of fractional differential equations." *J. of Mathematical Analysis and Applications* 334.1 (2007): 1-10. https://doi.org/10.1016/j.jmaa.2006.12.036
19. Batiha, Iqbal M., et al. "Existence and uniqueness of solutions for generalized Sturm–Liouville and Langevin equations via Caputo–Hadamard fractional-order operator." Engineering Computations 39.7 (2022): 2581-2603. https://doi:10.1108/EC-07-2021-0393

20. Park, Jong-An. "Mann-iteration process for the fixed point of strictly pseudo contractive mapping in some Banach spaces." *Journal of the Korean Mathematical Society* 31.3 (1994): 333-337.
21. Olaleru, Johnson O. "ON the Convergence Rates of Picard, Mann and Ishikawa iterations of generalized contractive operators." *Studia Universitatis Babes-Bolyai, Mathematica* 4 (2009).
22. Harjani, Jackie, Belen López, and Kishin Sadarangani "Existence and uniqueness of mild solutions for a fractional differential equation under Sturm-Liouville boundary conditions when the data function is of Lipschitzian type." *Demonstration Mathematica* 53.1 (2020): 167–173. https://doi.org/10.1515/dema-2020-0014
23. Furati, Khaled M., and Nasser-eddine Tatar. "An existence result for a nonlocal fractional differential problem." Journal of Fractional Calculus 26 (2004): 43-51.
24. Kou, Chunhai, Huacheng Zhou, and Changpin Li. "Existence and continuation theorems of fractional Riemann-Liouville type fractional differential equations." *International Jour. of Bifurcation and Chaos* 22.4 (2012). https://doi.org/10.1142/S0218127412500770
25. Klimek, Malgorzata, and Om P. Agrawal. "On a Regular Fractional Sturm-Liouville Problem with derivatives of order in (0,1)." *Proceedings of the 13th International Carpathian Control Conference (ICCC). IEEE*, (2012). doi:10.1109/CarpathianCC.2012.6228655
26. Klimek, Malgorzata, and Om Prakash Agrawal. "Fractional Sturm–Liouville problem." Comp. Math. Appl. 66.5(2013): 795-812. https://doi.org/10.1016/j.camwa.2012.12.011
27. Machado, J. Tenreiro, Virginia Kiryakova, and Francesco Mainardi. "Recent history of fractional calculus." Communications in nonlinear science and numerical simulation 16.3 (2011): 1140-1153. https://doi.org/10.1016/j.cnsns.2010.05.027
28. Ciesielski, Mariusz, Malgorzata Klimek, and Tomasz Blaszczyk. "The fractional Sturm–Liouville problem-Numerical approximation and application in fractional diffusion." J of Com. & App. Math. 317 (2017): 573–588. https://doi.org/10.1016/j.cam.2016.12.014
29. Klimek, Małgorzata, Agnieszka B. Malinowska, and Tatiana Odzijewicz. "Applications of the Fractional Sturm–Lioville problem to the Space–Time Fractional Diffusion in a finite Domain." *Fractional Calculus and Applied Analysis* 19.2 (2016): 516–550. https://doi:10.1515/fca-2016-0027
30. Klimek, Malgorzata, Mariusz Ciesielski, and Tomasz Blaszczyk. "Exact and Numerical Solutions of the Fractional Sturm–Liouville problem." *Fractional Calculus and Applied Analysis* 21.1 (2018): 45–7. doi: 10.1515/fca-2018-0004 .
31. Shahzad, Naseer, and Habtu Zegeye. "On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces." *Nonlinear Analysis: Theory, Methods & Applications* 71.3–4 (2009): 838 – 844.
32. Chipot, Michel. Handbook of Differential Equations: Stationary Partial Differential Equations. 6 (2008): 503-583. doi.org/10.1016/S1874-5733(08)80024-1
33. Popescu, Ovidiu. "Picard iteration converges faster than Mann iteration for a class of quasicontractive operators." *Math. Communications* 12.2 (2007): 195-202.
34. Podlubny, Igor. "Fractional Differential Equations." *Academic Press*, London (1999).
35. Podlubny, Igor. "Geometric and physical interpretation of fractional integration and fractional differentiation". Dedicated to the 60th anniversary of Prof. Francesco Mainardi. Fract. Calc. Appl. Anal. 5.4 (2002): 367-386. doi.org/10.48550/arXiv.math/0110241
36. Podlubny, Igor, et al. "Analogue realizations of fractional-order controllers." *Nonlinear Dynamic* 29(2002): 281-296.

37. Pandey, Prashant K., Rajesh K. Pandey, and Om P. Agrawal. "Variationalapproximation for Fractional Sturm–Liouville problem." *Fractional Calculus and Applied Analysis* 23 (2020): 861–874. doi: 10.1515/fca-2020-0043
38. Rivero, Margarita, Juan Trujillo, and M. Velasco. "A fractional approach to the Sturm-Liouville problem." *Open Physics* 11.10 (2013): 1246-1254. doi:10.2478/s11534-013-0216-2
39. Chang, Shih-shen, et al. "Iterative Approximations of Fixed Points and Solutions for Strongly Accretive and Strongly Pseudo-Contractive Mappings in Banach Spaces." *Journal of Mathematical Analysis and Applications* 224.1 (1998): 149 − 165.
40. Abbas, Syed. "Existence of Solutions to Fractional order ordinary and Delay differential equations and applications." *Electronic Journal of Differential Equations (EJDE)* [electronic only] 2011 (2011): 1–11.
41. Berinde, Vasile, and F. Takens. "Iterative Approximation of Fixed Points." *Iterative approximation of fixed points* 1912. Berlin: Springer (2007).
42. Berinde, Vasile. "Existence and approximation of Solutions of some first order iterative differential equations." *Miskolc Mathematic Notes* 11.1 (2010): 13–26.
43. Berinde, Vasile.; "Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators." *Fixed Point Theory & Applications* 2004.2 (2004): 97–105. dx.doi.org/10.1155/S1687182004311058
44. Mann, W. Robert. "Mean value methods in iteration." *Proceedings of the American Mathematical Society* 4.3 (1953): 506–510.
45. Wang, Wei-Chuan. "Some notes on conformable fractional Sturm–Liouville problems." *Boundary Value Problems* 2021, 103 (2021). doi.org/10.1186/s13661-021-01581-y
46. Yang, Xiong, Zhongli Wei, and Wei Dong. "Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations." *Communications in Nonlinear Science and Numerical Simulation* 17.1 (2012): 85–92.
47. Yu, Cheng, and Guozhu Gao. "Existence of fractional differential equations." *Journal of Mathematical Analysis and Applications* 310.1 (2005): 26-29.
48. Suzuki Tomonari. "Krasnoselskii and Mann's type sequences and Ishikawa's strong convergence theorem." *Proceedings of the Third International Conference on Nonlinear Analysis and Convex Analysis* (2004): 527-539.