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Fekete-Szegö inequality for Certain Subclass of Analytic Functions

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Abstract:

In this present work, the author obtain Fekete-Szegö inequality for certain classes of parabolic starlike and uniformly convex functions involving certain generalized derivative operator defined in [1].

1 Introduction

Let A denote the class of all analytic functions in the open unit disk

$$\mathrm{U}=\{z\in \mathrm{C}:|z|<1\},$$

and Let H be the class of functions f in A given by the normalized power series

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \quad (z\in \mathrm{U}). \tag{1}$$

Let S denote the class of functions which are univalent in U.

A function f in H is said to be uniformly convex in U if f is a univalent convex function along with the property that, for every circular arc γ contained in U, with centre γ also in U, the image curve $f(\gamma)$ is a convex arc.Therefore, the class of uniformly convex functions is denoted by UCV (see [3]).

It is a common fact from [12], [13] that, for $z\in \mathrm{U}$,that

$$f\in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right|<\Re\{1+\frac{zf''(z)}{f(z)}\} \quad ,(z\in \mathrm{U}). \tag{2}$$

Condition (2) implies that

$$1+\frac{zf''(z)}{f(z)},$$

lies in the interior of the parabolic region

$$R:=\{w:w=u+iv \quad and \quad v^2<2u-1\},$$

for every value of $z \in \mathrm{U}$. Let

$$P := \{p : p \in A; p(0) = 1 \quad and \quad \Re(p(z)) > 0; \quad z \in \mathrm{U}\},$$

and

$$PAR := \{p : p \in P \quad and \quad p(\mathrm{U}) \subset R\}.$$

A function f in H is said to be in the class of parabolic starlike functions, denoted by SP (cf. [13]), if

$$\frac{zf''(z)}{f'(z)} \in R \quad ,(z \in \mathrm{U}).$$

Let the functions f given by (1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \ (z \in \mathrm{U}),$$

then the Hadamard product (convolution) of f and g, defined by :

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \ (z \in \mathrm{U}).$$

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & for \quad k = 0, \\ x(x+1)(x+2)...(x+k-1) & for \quad k \in \mathrm{N} = \{1,2,3,...\}. \end{cases}$$

The authors in [1] have recently introduced a new generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ as the following:

to derive our generalized derivative operator, we define the analytic function

$$\varphi^m(\lambda_1, \lambda_2, l)(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} z^k, \tag{3}$$

where $m \in \mathrm{N}_0 = \{0,1,2,...\}$ and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0$.

Definition 1 *For $f \in A$ the operator $I^m(\lambda_1, \lambda_2, l, n)$ is defined by* $I^m(\lambda_1, \lambda_2, l, n): A \to A$

$$I^m(\lambda_1, \lambda_2, l, n)f(z) = \varphi^m(\lambda_1, \lambda_2, l)(z) * R^n f(z), \qquad (z \in \mathrm{U}), \tag{4}$$

where $m \in \mathrm{N}_0 = \{0,1,2,...\}$ and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0,$ and $R^n f(z)$ denotes the Ruscheweyh derivative operator [4], and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k, (n \in \mathrm{N}_0, z \in \mathrm{U}),$$

where $c(n,k) = \dfrac{(n+1)_{k-1}}{(1)_{k-1}}$.

If $f \in H$, then the generalized derivative operator is defined by

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k) a_k z^k,$$

where $n,m \in \mathrm{N}_0 = \{0,1,2,...\}$, and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0, c(n,k) = \dfrac{(n+1)_{k-1}}{(1)_{k-1}}$.

Special cases of this operator includes:

- the Ruscheweyh derivative operator [4] in the cases:

$$I^1(\lambda_1,0,l,n) \equiv I^1(\lambda_1,0,0,n) \equiv I^1(0,0,l,n) \equiv I^0(0,\lambda_2,0,n)$$

$$\equiv I^0(0,0,0,n) \equiv I^{m+1}(0,0,l,n) \equiv I^{m+1}(0,0,0,n) \equiv R^n,$$

- the S$\hat{a}$ l$\hat{a}$ gean derivative operator [5]: $I^{m+1}(1,0,0,0) \equiv S^n$,
- the generalized Ruscheweyh derivative operator [6]: $I^2(\lambda_1,0,0,n) \equiv R_\lambda^n$,
- the generalized S$\hat{a}$ l$\hat{a}$ gean derivative operator introduced by Al-Oboudi [7]: $I^{m+1}(\lambda_1,0,0,0) \equiv S_\beta^n$,
- the generalized Al-Shaqsi and Darus derivative operator[8]: $I^{m+1}(\lambda_1,0,0,n) \equiv D_{\lambda,\beta}^n$,
- the Al-Abbadi and Darus generalized derivative operator [9]: $I^m(\lambda_1,\lambda_2,0,n) \equiv \mu_{\lambda_1,\lambda_2}^{n,m}$,

and finally

- the Catas drivative operator [10]: $I^m(\lambda_1,0,l,n) \equiv I^m(\lambda,\beta,l)$.

Using simple computation one obtains the next result.

$$(l+1)I^{m+1}(\lambda_1,\lambda_2,l,n)f(z) = (1+l-\lambda_1)[I^m(\lambda_1,\lambda_2,l,n) * \varphi^1(\lambda_1,\lambda_2,l)(z)]f(z) +$$

$$\lambda_1 z[(I^m(\lambda_1,\lambda_2,l,n)*\varphi^1(\lambda_1,\lambda_2,l)(z)]'. \tag{5}$$

Where $(z\in \mathrm{U})$ and $\varphi^1(\lambda_1,\lambda_2,l)(z)$ analytic function and from (3) given by

$$\varphi^1(\lambda_1,\lambda_2,l)(z)=z+\sum_{k=2}^{\infty}\frac{1}{(1+\lambda_2(k-1))}z^k.$$

Definition 2 *Let* $SP^m(\lambda_1,\lambda_2,l,n)$ *be the class of functions* $f\in H$ *satisfying the inequality :*

$$\left|\frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}-1\right|<\Re\left\{\frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right\},\quad (z\in \mathrm{U}). \tag{6}$$

Note that many other operators are studied by many different authors, see for example [19, 20, 21]. There are times, functions are associated with linear operators and create new classes (see for example [18]). Many results are considered with numerous properties are solved and obtained.

However, in this work we will give sharp upper bounds for the Fekete-Szegö problem. It is well known that Fekete and Szegö [14] obtained sharp upper bounds for $|a_3-\mu a_2^2|$ for the case $f\in S$ and μ is real. The bounds have been studied by many since the last two decades and the problems are still being popular among the writers. For different subclasses of S, the Fekete-Szegö problem has been investigated by many authors including [14, 12, 15, 16, 17], few to list. For a brief history of the Fekete-Szegö problem see [17].In the present paper we completely solved the Fekete-Szegö problem for the class $SP^m(\lambda_1,\lambda_2,l,n)$ defined by using $I^m(\lambda_1,\lambda_2,l,n)$.

2 Fekete-Szegö problem for the class $SP^m(\lambda_1,\lambda_2,l,n)$

Here we obtain sharp upper bounds for the Fekete-Szegö functional $|a_3-\mu a_2^2|$ for functions f belonging to the class $SP^m(\lambda_1,\lambda_2,l,n)$,

Let the function f, given by

$$f(z)=z+a_2z^2+a_3z^3+\ldots,(z\in \mathrm{U}), \tag{7}$$

be in the class $SP^m(\lambda_1,\lambda_2,l,n)$. Then by geometric interpretation there exists a function w satisfying the conditions of the Schwarz ' lemma such that

$$\frac{z(I^m(\lambda_1,\lambda_2,l,n)'f(z))}{I^m(\lambda_1,\lambda_2,l,n)f(z)}=q(w(z)),\quad (z\in \mathrm{U}).$$

It can be verified that the Riemann map q of U onto the region R, satisfying $q(0)=1$ and $q_0(0)>0$, is given by

$$q(z)=1+\frac{2}{\pi^2}(\log\frac{1+\sqrt{z}}{1-\sqrt{z}})^2,$$

$$=1+\frac{8}{\pi^2}\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{2k+1})z^n,$$

$$=1+\frac{8}{\pi^2}(z+\frac{2}{3}z^2+\frac{23}{45}z^3+\frac{44}{105}z^4+\ldots), \qquad (z\in \mathrm{U}).$$

Let the function P_1 in P be defined by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots, \qquad (z\in \mathrm{U}).$$

Then by using

$$w(z)=\frac{p_1(z)-1}{p_1(z)+1},$$

we obtain

$$a_2=\frac{4(1+l)^{m-1}(1+\lambda_2)^m}{\pi^2(1+\lambda_1+l)^{m-1}(n+1)}c_1,$$

and

$$a_3=\frac{4(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}(c_2-\frac{c_1^2}{6}(1-\frac{24}{\pi^2})).$$

These expressions shall be used throughout the rest of the paper.

In order to prove our result we have to recall the following lemmas:

Lemma 1 *[11] If* $p_1(z)=1+c_1z+c_2z^2+\cdots$ *is an analytic function with positive real part in* U, *then*

$$|c_2-\upsilon c_1^2|\le\begin{cases}-4\upsilon+2 & \text{if} \quad \upsilon\le 0,\\ 2 & \text{if} \quad 0\le\upsilon\le 1,\\ 4\upsilon+2 & \text{if} \quad \upsilon\ge 1.\end{cases}$$

When $\upsilon < 0$ or $\upsilon > 1$, the equality holds if and only if $p_1(z)$ is $\frac{(1+z)}{(1-z)}$ or one of its rotations. If $0 < \upsilon < 1$, then the equality holds if and only if $p_1(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\upsilon = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}a\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}a\right)\frac{1-z}{1+z} \quad (0 \le a < 1),$$

or one of its rotations. If $\upsilon = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\upsilon = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < \upsilon < 1$:

$$|c_2 - \upsilon c_1^2| + \upsilon |c_1| \le 2, \qquad (0 < \upsilon \le \frac{1}{2}),$$

and

$$|c_2 - \upsilon c_1^2| + (1-\upsilon)|c_1| \le 2, \qquad (\frac{1}{2} < \upsilon \le 1).$$

Lemma 2 *[2] Let* h *be analytic in* U *with* $\Re\{h(z)\} > 0$ *and be given by* $h(z) = 1 + c_1 z + c_2 z^2 + \ldots,$ *for* $z \in \mathrm{U}$, *then*

$$|c_2 - \frac{c_1^2}{2}| \le 2 - \frac{|c_1|^2}{2}.$$

Lemma 3 *[11] Let* $h \in P$ *where* $h(z) = 1 + c_1 z + c_2 z^2 + \cdots.$

Then $$|c_n| \le 2, \quad n \in \mathrm{N},$$

and $$|c_2 - \frac{1}{2}\mu c_1^2| \le 2 + \frac{1}{2}(|\mu - 1| - 1)|c_1|^2.$$

Theorem 1 *If* f *be given by (1) and belongs to the class* $SP^m(\lambda_1, \lambda_2, l, n)$. *Then,* $|a_3 - \mu a_2^2| \le$

$$\begin{cases} \dfrac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[\dfrac{4\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1(k-1)+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\dfrac{1}{3}-\dfrac{4}{\pi^2}\right] & if \quad \mu\le\sigma_1, \\ \dfrac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)} & if \quad \sigma_1\le\mu\ge\sigma_2, \\ \dfrac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[\dfrac{1}{3}+\dfrac{4}{\pi^2}-\dfrac{4\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1(k-1)+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right] & if \quad \mu\le\sigma_2, \end{cases} \quad (8)$$

where

$$\sigma_1=\frac{(1+2\lambda_2)^m(1+\lambda_1+l)^{2(m-1)}(n+1)}{(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)^{m-1}(n+2)}\left(1+\frac{5\pi^2}{24}\right), \quad (9)$$

$$\sigma_2=\frac{(1+2\lambda_2)^m(1+\lambda_1+l)^{2(m-1)}(n+1)}{(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)^{m-1}(n+2)}\left(1-\frac{\pi^2}{24}\right). \quad (10)$$

each of the estimates in (8) is sharp.

Proof: An easy computation shows that

$$|a_3-\mu a_2^2|=\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left|\left(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{1}{3}-\frac{8}{\pi^2}\right)c_1^2-2c_2\right| \quad (11)$$

$$\le\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left[\left(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{5}{3}-\frac{8}{\pi^2}\right)|c_1|^2+2|c_1^2-c_2|\right]. \quad (12)$$

If $\mu\ge\sigma_1$, then the expression inside the first modulus on the right-hand side of (12) is nonnegative.

Thus,by applying Lemma 3 ,we get

$$=\frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left[\left(\frac{4\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{1}{3}-\frac{4}{\pi^2}\right)\right], \qquad (13)$$

which is the assertion (8). Equality in (13) holds true if and only if $|c_1|=2.$ Thus the function f is $k(z;0;1)$ or one of its rotations for $\mu>\sigma_1.$

Next, if $\mu\le\sigma_2,$ then we rewrite (11) as

$$|a_3-\mu a_2^2|=\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left|\left(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}+\frac{1}{3}-\frac{8}{\pi^2}\right)c_1^2-2c_2\right|$$

$$\le\frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left|\left(\frac{1}{3}+\frac{8}{\pi^2}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right)\right|.$$

The estimates $|c_2|\le 2$ and $|c_1|\le 2,$ after simplification, yield the second part of the assertion (8), in which equality holds true if and only if f is a rotation of $k(z;0;1)$ for $\mu<\sigma_2,.$ If $\mu=\sigma_2,$, then equality holds true if and only if $|c_2|=2.$ In this case, we have

$$p_1(z)=\left(\frac{1+\nu}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\nu}{2}\right)\frac{1-z}{1+z} \qquad (0\le\nu<1; z\in \mathrm{U}).$$

Therefore the extremal function f is $k(z;0;\nu)$ or one of its rotations.

Similarly, $\mu=\sigma_1,$ is equivalent to

$$\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{5}{3}-\frac{8}{\pi^2}=0.$$

Therefore, equality holds true if and only if $|c_1^2-c_2|=2.$

This happens if and only if

$$\frac{1}{p_1(z)}=\left(\frac{1+\nu}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\nu}{2}\right)\frac{1-z}{1+z} \qquad ,(0\le\nu<1; z\in \mathrm{U}).$$

Thus the extremal function f is $k(z;\pi;\nu)$ or one of its rotations.

Finally, we see

$$|a_3-\mu a_2^2|=\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left|2(c_2-\frac{1}{2}c_1^2)+\left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right)c_1^2\right|,$$

and

$$\max\left|\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right|\le 1, \qquad (\sigma_1\le\mu\ge\sigma_2).$$

Therefore, using Lemma 3, we get

$$|a_3-\mu a_2^2|\le\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[2(2-\frac{1}{2}|c_1|^2)+|c_1|^2\right]$$

$$=\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)} \qquad ,if \qquad \sigma_1\le\mu\ge\sigma_2.$$

If $\sigma_1<\mu<\sigma_2$, then equality holds true if and only if $|c_1|=0$ and $|c_2|=0$. Equivalently, we have

$$p_1(z)=\frac{1+\nu z^2}{1-\nu z^2} \quad ,(0\le\nu\le 1; z\in \mathrm{U}).$$

Thus the extremal function f is $k(z;0;0)$ or one of its rotations.

3 IMPROVEMENT OF THE ESTIMATION

Theorem 2 *If $\sigma_1\le\mu\le\sigma_2$, then in view of Lemma , Theorem can be improved as follows:*

$$|a_3-\mu a_2^2|+\left(\mu-\frac{(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}{(1+l)^{m-1}(1+\lambda_2)^m(1+2\lambda_1+l)^{m-1}(n+2)}(1-\frac{\pi^2}{24})\right)|a_2|^2$$

$$\le\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1(k-1)+l)^{m-1}(n+1)(n+2)} \quad (\sigma_2\le\mu\le\sigma_3),$$

and

$$|a_3-\mu a_2^2|+\left(\frac{(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}{(1+l)^{m-1}(1+\lambda_2)^m(1+2\lambda_1+l)^{m-1}(n+2)}(1+\frac{5\pi^2}{24})-\mu\right)|a_2|^2$$

$$\leq\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1(k-1)+l)^{m-1}(n+1)(n+2)}\quad(\sigma_3\leq\mu\leq\sigma_1),$$

where σ_1 and σ_2 are given, as before, by (9), (10), and

$$\sigma_3=\frac{(1+2\lambda_2)^m(1+\lambda_1+l)^{2(m-1)}(n+1)}{(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)^{m-1}(n+2)}(1+\frac{\pi^2}{12}).$$

Proof: For the values of $\sigma_1\leq\mu\leq\sigma_3$, and from Lemma 2 we have

$$|a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2|^2\leq$$

$$\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[2(2-\frac{1}{2}|c_1|^2)+\left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right)|c_1|^2\right]$$

$$+\frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^4(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[\frac{\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(n+2)}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)}-(1-\frac{\pi^2}{24})\right]|c_1|^2$$

$$\leq\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}[4-|c_1|^2+\left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right)|c_1|^2$$
$$+\left(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{8}{\pi^2}+\frac{1}{3}\right)|c_1|^2]$$

$$=\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}.$$

Similarly, if $\sigma_2\leq\mu\leq\sigma_3$, we can write

$$|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2 \le \frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left[2(2-\frac{1}{2}|c_1|^2)+(\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2(k-1))^m(n+2)}{\pi^2(1+\lambda_1(k-1)+l)^{m-1}(n+1)})|c_1|^2\right]$$

$$+\frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^4(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[1+\frac{5\pi^2}{24}-\frac{\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right]|c_1|^2$$

$$\le \frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}[4-|c_1|^2+(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{8}{\pi^2}-\frac{2}{3})|c_1|^2$$

$$+(\frac{8}{\pi^2}+\frac{5}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)})|c_1|^2]$$

$$=\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}.$$

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