COMPUTATION OF HYPERGEOMETRIC FUNCTIONS

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Abstract

This research explored some methods for computing the hypergeometric function which can in some cases be difficult to find quickly and accurately. It has found that some softwares, such as *Maple*, are of little use in such instances. So, in this case, this research highlights a method to compute a special case of the hypergeometric function which is $\Phi(a,c,z)=\Phi(\frac{3}{4}-\frac{it}{2},\frac{3}{2},\frac{i\pi\alpha^2 x}{4})$ in a very fast time compared with the *Maple*.

Keywords: hypergeometric function.

Introduction

The calculations of the hypergeometric function ${}_pF_q$ of mathematical physics are often required in many branches of applied mathematics. Despite the importance of this topic, this is sometimes a very hard in practice. The main reason for this is that the function has the complicated structure which produces many interesting mathematical intricacies. The research will focus on computing one of commonly used hypergeometric functions which is ${}_1F_1(a;c;z)=\Phi(a;c;z),$ which is also called Kummer's function as discussed by Abramowitz and Stegun (1972) [1]. Then, It will define the saddle point of a contour integral because it will help to improve the method of solution. In the next section, the research discusses estimation the confluent hypergeometric function using saddle point. The *Maple* software will be used to compute this function. Therefore, the aim of this research is to find a quick and easy method in which we can calculate the values of this function which is a reliable for many different variables. We shall be especially concerned with the case when the magnitude of a and z are large, when special asymptotic formulas have to be developed.

1 Asymptotic expansion of integrals

1.1 The confluent hypergeometric (Kummer's function)

There are many functions defined as special cases of the general confluent hypergeometric function ${}_1F_1(a;b;z)$, including the incomplete gamma function, Modified Bessel functions and Laguerre polynomials as they are suggested by Abramowitz and Stegun (1972) [1]. The research shall be investigating the regular solution (at $z=0$) often denoted by $M(a,b,z)$ (as opposed to $U(a,b,z)$ the irregular solution) of Kummer's differential equation.

$$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}+aw=0. \quad (1)$$

Formally $M(a,b,z)$ is defined by 1 where $M(a,b,z)$ is given by

$$M(a,b,z)=\sum_{s=0}^{\infty}\frac{(a)_s}{(b)_s s!}z^s$$

$$=1+\frac{a}{b}z+\frac{a(a+1)}{b(b+1)2!}z^2+\ldots \quad (2)$$

where a,b and $z \in \mathbb{C}$ $(b \neq 0,-1,-2,-3,\ldots,-n\,;n \in \mathbb{N})$ which is detailed in [1].

The target of this research is not just to use the series 2, but also to find a method to compute this function using *Maple* without using the intrinsic function of Maple ($hypergeom([a],[b],z)$).This special function in *Maple* can calculate the hypergeometric functions in some cases, but sometimes we find that the *Maple* routine is very slow. This is particularly true when $|a|$ or $|z|$ are large compared to $|b|$. However, to begin research shall consider how $M(a,b,z)$ can be calculated directly using the series 2. This series is solved by writing code in *Maple* which calculates the value of the series from $s=0$ to n and termination occurs when *Maple* gets a very small relative error.

1.2 Example

Compute the hypergeometric function using the series 2 if we have:

(i). $a=150I, b=166.0$ and $z=1.1I$

(ii). $a=15000I, b=166.0$ and $z=10000.1I$.

(I means that the number is complex number).

where accurate to a relative error of $\varepsilon = 0.00000001$ by:

(a) Method of Series (Code **A**).

(b) Using Maple's intrinsic function, and then compare the solution in both cases.

First case(i): by the Code **A**, research has got that the value of the confluent hypergeometric function $= [0.369003792014086 - 0.00122244630148484I]$ where $n = 12$ which means that it has taken 12 steps to find the solution with this relative error. By using the intrinsic function in *Mapli. Hypergeom*([150*I*], [166.0], 1.1I) = $[0.369003791936315 - 0.00122244625750552I]$. If we compare the two solutions, it is clearly that the relative error quite low which is equal just $-4.4\times10^{-11}I$.

In the **second case (ii),** the research had found that the solution can not be found by the series method because the condition $\left[\frac{|a|\times|z|}{|b|}\right] \geq 1$ has not been achieved in the code (Code **A**). While Maple have taken more than two minutes (exactly $132.63s$) to calculate the hypergeometric function. For this reason, research will find another method to compute the hypergeometric function.

2 Saddle points and method steepest descent

Consider the following integral:

$$I(t) = \int_a^b g(x)e^{it\phi(x)}dx. \qquad (3)$$

Here a and $b \in \mathbb{R}$, $a < b$, $t \in \mathbb{R}, t > 1$, and $g(x), \phi(x)$ are real valued ($g(x)$ could be $\in \mathbb{C}$) functions with $g(x)$ defined as that $\int_a^b |g(x)|dx < \infty$ (basically the integral $I(t)$ will exist). Also, $g(x)$ does not contain an exponential term. What research need is a way of obtaining a quick and easy estimate for $I(t)$, with an error term which declines quickly as t gets large. This can be done using the Saddle Point Method of steepest descent, as discussed by Bender and Orszag (1999) [2].

2.1 Saddle points

The first step is to replace x by $z \in \mathbb{C}$ and consider $I(t)$ as a complex integral around some suitable complex contour C.

$$I(t) = \int_C g(z)e^{it\phi(z)}dz.$$

Now assume $\phi(z)$ is a well behaved multi-differentiable, analytic function of $\mathbb{C}$. Applying Taylor's Theorem about a point $z = z_0$ gives :

$$\phi(z) = \phi(z_0) + (z - z_0)\phi'(z_0) + \frac{(z - z_0)^2}{2}\phi''(z_0) + O(z - z_0)^3. \qquad (4)$$

A saddle point of a complex function $\phi(z)$ is a point $z = z_0$ where $\phi'(z_0) = 0$. Suppose $z = z_0$ is saddle point. Then near $z = z_0$ use integral behaves like:

$$I(t) \approx \int_{near z_0} g(z) e^{it[\phi(z_0) + \frac{(z-z_0)^2}{2}\phi''(z_0)]} dz. \quad (5)$$

The problem with our original integral:

$$I(t) = \int_a^b g(x) e^{it\phi(x)} dx$$

$$= \int_a^b \left(\cos[t\phi(x)] + i\sin[t\phi(x)]\right) g(x) dx, \quad (6)$$

is that as $t \to \infty$ the $e^{it\phi(x)}$ oscillates faster and faster but does not get any smaller. This means that we have to consider integrating over the whole interual to $x \in [a,b]$ to estimate $I(t)$. If $[a,b]$ is large, such as if $b \to \infty$ this will be a lengthy process, even for a computer. In the case $[a,\infty)$, we also have to worry about how $g(x) \to 0$ as $b \to \infty$ to ensure the integral converges. This may happen quite slowly.

The advantage of the saddle point (steepest descent method) is that it localizes the behaviour of the integral around $z = z_0$ (Bender and Orszag 1999)[2]. This makes the integral much easier to estimate. To do that we will consider:

$$I(t) \approx e^{it\phi(z_0)} \int_{near z_0} g(z) e^{\frac{it(z-z_0)^2}{2}\phi''(z_0)} dz, \quad (7)$$

and assume $\phi''(z_0) > 0$ (the case $\phi''(z_0) < 0$ is easy to deal with change $\frac{\pi}{4}$ to $\frac{-\pi}{4}$ in what follows). To make things easier, assume $z_0 \in \mathbb{R}$ and let $C_1 =$ path in the complex plane (see figure 2.1) such that: $C_1(z) \Rightarrow z = z_0 + \delta e^{i\frac{\pi}{4}}, \delta \in (-R, R)$.

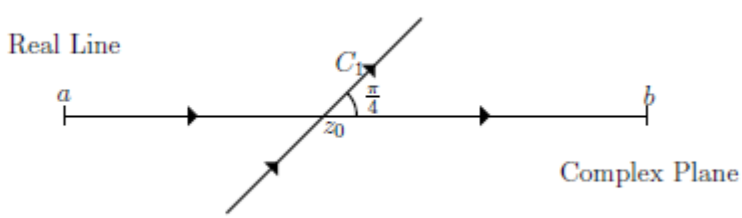

Figure 2.1: The graph of real line and complex plane.

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Clearly this path pases through $z = z_0$ when $\delta = 0$. Along this path we have:

$$g(z)e^{\frac{it(z-z_0)^2}{2}\phi''(z_0)} = g(z_0 + \delta e^{i\frac{\pi}{4}})e^{it(i\delta^2)\frac{\phi''(z_0)}{2}}$$

$$= g(z_0 + \delta e^{i\frac{\pi}{4}})e^{-t\delta^2\frac{\phi''(z_0)}{2}}. \qquad (8)$$

Therefore, the integral for C_1 will be:

$$I(t) = \int_{C_1} g(z)e^{it\phi(z)}dz$$

$$= e^{i\frac{\pi}{4}+it\phi(z_0)}\int_{-R}^{R} g(z_0 + \delta e^{i\frac{\pi}{4}})e^{-t\delta^2\frac{\phi''(z_0)}{2}}d\delta \qquad (9)$$

The function $e^{-t\delta^2\frac{\phi''(z_0)}{2}}$ dies off very rapidly as $|\delta|$ gets large. This means that the $\int_{C_1} dz$ is localised to the area very close to $z = z_0$. The path $z = z_0 + \delta e^{i\frac{\pi}{4}}$ is called Path of Steepest Descant, and along this path, the function $e^{\frac{it(z-z_0)^2}{2}\phi''(z_0)}$ dies off most rapidly. Such steepest descent paths are always associated with saddle points. So, the integral $I(t)$ along C_1 is given by

$$I(t) = e^{i[\frac{\pi}{4}+t\phi(z_0)]}\int_{-R}^{R}\sum_{k=0}^{N}\frac{g^{(k)}(z_0)[\delta e^{i\frac{\pi}{4}}]^k}{k!}e^{-t\delta^2\frac{\phi''(z_0)}{2}}d\delta$$

$$\approx e^{i[\frac{\pi}{4}+t\phi(z_0)]}\left[\frac{\sqrt{\pi}}{\frac{t\phi''(z_0)^{\frac{1}{2}}}{\sqrt{2}}}g(z_0) + \frac{1!!\sqrt{\pi}\,g''(z_0)[e^{i\frac{\pi}{4}}]^2}{\frac{2!(t\phi''(z_0))^{\frac{3}{2}}}{\sqrt{2}}} + \frac{3!!\sqrt{\pi}\,g^{(4)}(z_0)[e^{i\frac{\pi}{4}}]^4}{\frac{4!(t\phi''(z_0))^{\frac{5}{2}}}{\sqrt{2}}} + \ldots\right]$$

$$\approx \sqrt{\pi}\;e^{i[\frac{\pi}{4}+t\phi(z_0)]}\sum_{k=0}^{N}\left[\frac{(2k-1)!![t\phi''(z_0)]^{2k}g^{(2k)}(z_0)}{\frac{(2k)![t\phi''(z_0)]^{(2k-1)/2}}{\sqrt{2}}}\right] \qquad (10)$$

This is because we have the fact which is:

$$\int_{-\infty}^{\infty} x^{2n+k} e^{-px^2} dx = \begin{cases} \frac{(2n-1)!!}{(2p)^n}\sqrt{\frac{\pi}{p}} & \text{if } \ k=0, \\ 0 & \text{if } \ k=1. \end{cases} \qquad (11)$$

Provided $[t\phi''(z_0)/2]$ is large, the series 10 will consist initially of rapidly decreasing terms the value of N is used to truncate the series when the terms start to increase. Hence, to first order the integral 9 along C_1 will be

$$I(t); e^{i[\frac{\pi}{4}+t\phi(z_0)]} g(z_0)\sqrt{\frac{\pi}{t\phi''(z_0)/2}} \qquad (12)$$

2.2 Contour Integrals

The integral 3 can be solved by using the saddle point, as stated in [2, 3]. This will be done by forming the path of integration as in the figure 2.2.

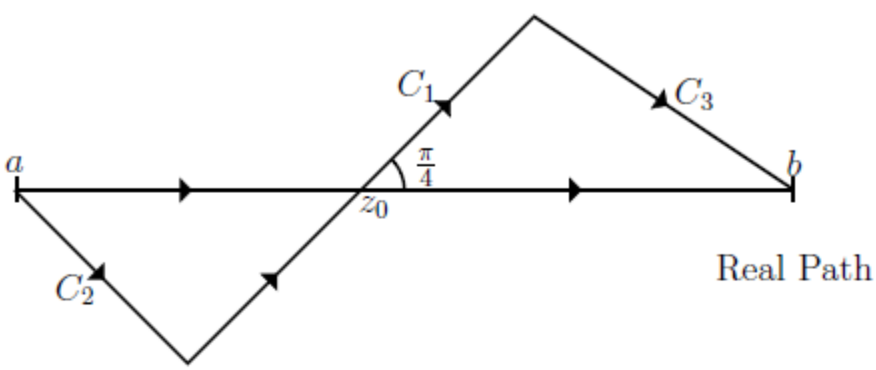

Figure 2.2: Contour of integration in equation 14.

Assuming the function $g(z)$ has no singularities, we can invoke Cauchy's Theorem. Thus:

$$\int_a^b - \int_{C_1} - \int_{C_2} - \int_{C_3} = 0. \qquad (13)$$

Therefore:

$$I(t)=\int_{C_1} g(z)e^{it\phi(z)}dz+\int_{C_2} g(z)e^{it\phi(z)}dz+\int_{C_3} g(z)e^{it\phi(z)}dz \qquad (14)$$

Typically the integrals C_2 and C_3 are smaller than the integral along C_1 because they do not pass through any saddle points of $\phi(z)$. In which case

$$\int_{C2} \text{ and } \int_{C3} \sim \frac{1}{t} \text{ compared to } \int_{C_1} \sim \frac{1}{\sqrt{t}}$$

$$\Rightarrow I(t)=\int_a^b g(x)e^{it\phi(x)}dx \approx e^{i[\pi/4+t\phi(z_0)]}g(z_0)\sqrt{\frac{\pi}{t\phi''(z_0)/2}}+O\left(\frac{1}{t}\right). \qquad (15)$$

2.3 Example

Suppose one has the integral

$$\int_0^{100} \frac{e^{it(x^2-2x)}}{x^2+1}dx \qquad \text{for } t=10,100,1000,\ldots$$

Here $\phi(x)=x^2-2x$, and $g(x)=\dfrac{1}{x^2+1}$ which is well behaved, and dies off sufficiently rapidly for this integral to exist. So this integral will be estimated by using the saddle point method.

$$\phi(x)=x^2-2x \Rightarrow \phi'(x)=2x-2$$

$$\Rightarrow 2x-2=0 \Rightarrow x=1$$

$$\Rightarrow z_0=1 \text{ is the saddle point}$$

$$\Rightarrow \phi''(z_0)=2$$

So near $z_0=1$ using Taylor's Theorem 4 gives $\phi(z)=-1+(z-1)^2$. Set up a steepest descent path $C_1(z)\mapsto z=1+\delta e^{i\frac{\pi}{4}}$ through $z=1$. Then near $z=1$

$$\Rightarrow g(z)=\frac{1}{x^2+1}\sim\frac{1}{2} \text{ so along } C_1$$

$$\Rightarrow \int_{C_1} \frac{e^{it(x^2-2x)}}{x^2+1}dx \approx \frac{1}{2}e^{-it}\int_{-R}^{R} e^{-it\delta^2+\frac{i\pi}{4}}d\delta.$$

Finally, the result for the integral $I(t)$ will be

$$I(t) \approx \frac{\sqrt{\pi}\ e^{-it+\frac{i\sqrt{\pi}}{4}}}{2\sqrt{t}}. \qquad (16)$$

Now the result 16 will be used to compute the hypergeometric function 2 from $t=10$ to 40. The *Maple* has used to compute the result 16 to find the relative error in the approximation from $t=10$ to 40 between the result 16 and the solution by integration directly in *Maple*. In this case, *Maple* has found the integration hard and had taken a long time to compute it when t got large. On the other hand, with the result 16, Maple has computed the integral much faster, as will be shown. If we run the code **B** in *Maple*, we will get the result as in the table 1 which shows some values from that result. Note that T means calculate the time difference between the time for the first solution by the result 16 and the time for the second solution by the integration directory in *Maple*, and ε is the percentage of error between the two solutions.

There are some special cases that have gotten in the table 1. For example, the biggest relative error was when $t=11$. Then, the percent of error decreased step by step until $t=25$ where in this value of t, *Maple* has taken the biggest time to compute the the hypergeometric function ($T=11.123_s$). Suddenly, *Maple* can not compute the hypergeometric function when $t = 26, 27$ and 28 (where the result 14 has computed these values very fast). Therefore, some testes have made for these values to make *Maple* computing them as shown in the code **B**. In another case, the smallest relative error was when $t=29$. After that, *Maple* continues to compute the values until $t=40$.

The figure 2.3 shows us the values of $[t]$ with the time $[T]$ which is clearly that the time for computing the hypergeometric function has increased as t gets large, where the figure 2.4 shows that the relative error has decreased as t gets large.

Table 1: The table of some values of t with times $[T]$ and relative error $[\varepsilon]$.

t	Time taken (T_s)(seconds)	ε
10	2.777	18.96956394%
11 ⋮	4.212 ⋮	20.73015085% ⋮
25	11.123	12.21783312%
26 ⋮	5.585 ⋮	9.981694228% ⋮
29 ⋮	6.209 ⋮	0.1729691650% ⋮
40	6.006	7.007588679%

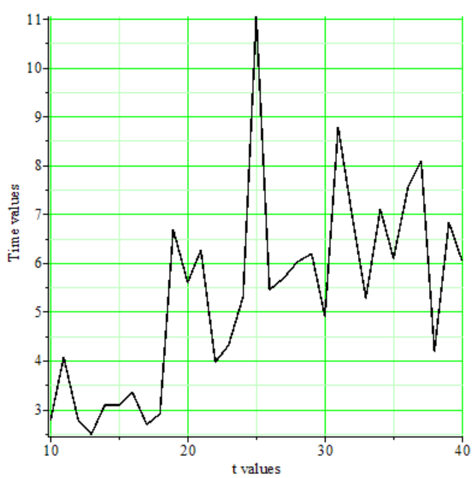

Figure 2.3: Values of $[t]$ With Times $[T]$.

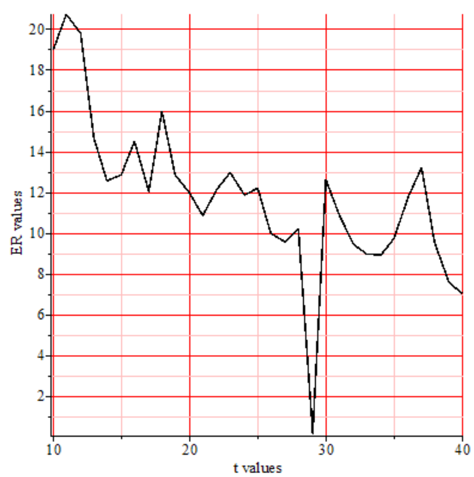

Figure 2.4: Values of $[t]$ with relative error ε.

However, if the code **B** will run with values of t bigger than 40 (for example $t = 60$), *Maple* may not be able to compute the integral. Therefore, section 3 will find another method to compute the hypergeometric function for large values of t.

3 Estimating the Confluent Hypergeometric Function Using Saddle Points

3.1 The analog of Euler's formula

The confluent hyper-geometric function $\Phi(a,c,z)=\Phi(\frac{3}{4}-\frac{it}{2},\frac{3}{2},\frac{i\pi\alpha^2 x}{4})$ plays an important role in the evaluation of the Hardy function $Z(t)$ which gives the amplitude of Riemann's zeta function along the critical line $z=\frac{1}{2}+it$.

Let us look how we might evaluate the more general function $\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4})$. Here both of $t\in\mathbb{R}$ and $\alpha\in\mathbb{R}$ are assumed to be large parameters and also $|t|\gg q, |\alpha|\gg q, q\in\mathbb{R}$ and $q>0$ (for simplicity we can assume $t>0$ and $t<0$ is a simple generalisation). The series representation of

$$\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4})=\sum_{s=0}^{\infty}\frac{(\frac{q}{2}-\frac{it}{2})_n(\frac{i\pi\alpha^2 x}{4})^n}{(q)_s s!} \qquad (17)$$

is totally useless. It would take millions of terms to even make the number start to get smaller if both t and α are large. Consider some other method which will be Euler's integral formula for the confluent hyper-geometric function and it is given by:

$$\Phi(a,c,z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{zx}x^{(a-1)}(1-x)^{(c-a-1)}dx$$

where $Re(c)>Re(a)>0$ and $Re(c)=q$ and $Re(a)=\frac{q}{2}$ which is detailed in [2, 3]. Hence

$$\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4})$$

$$=\frac{\Gamma(q)}{\Gamma(\frac{q}{2}-\frac{it}{2})\Gamma(\frac{q}{2}+\frac{it}{2})}\int_0^1 e^{\frac{i\pi\alpha^2 x}{4}}x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx \qquad (18)$$

The respective Gamma functions are well understood and can be calculated quite easily in *Maple*, using the famous Sterling's series for $\Gamma(z)$, which is valid for large $|z|$. So the key to find an estimate for $\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4})$ is the integral:

$$\int_0^1 e^{\frac{i\pi\alpha^2 x}{4}} x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx \qquad (19)$$

Firstly, let us split up the range of the integral into two parts, $x \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. Then for the second integral make the substitution,

$$y=(1-x) \Rightarrow x=1-y \Rightarrow dx=-dy \qquad (20)$$

$$\Rightarrow \int_{\frac{1}{2}}^1 e^{\frac{i\pi\alpha^2 x}{4}} x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx$$

$$= \int_{\frac{1}{2}}^0 e^{\frac{i\pi\alpha^2(1-y)}{4}} (1-y)^{(\frac{q}{2}-\frac{it}{2}-1)} y^{(\frac{q}{2}+\frac{it}{2}-1)}(-dy)$$

$$= e^{\frac{i\pi\alpha^2}{4}} \int_0^{\frac{1}{2}} e^{\frac{i\pi\alpha^2(-y)}{4}} (1-y)^{(\frac{q}{2}-\frac{it}{2}-1)} y^{(\frac{q}{2}+\frac{it}{2}-1)}dy. \qquad (21)$$

This means that the integral 19 equals same integral 20 for $x \in (0,\frac{1}{2})$ plus $e^{(\frac{i\pi\alpha^2}{4})}$ multiplied by complex conjugate of same integral for $x \in (0,\frac{1}{2})$. So we just need to know $\int_0^{\frac{1}{2}} e^{\frac{i\pi\alpha^2 x}{4}} x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx$ (since we can easily get its complex conjugate).

Now make a further substitution $x=\frac{1}{w+1}$. This gives us

$$dx = \frac{-1}{(w+1)^2}dw; \qquad w \in (1,\infty)$$

and $\quad 1-x=\frac{w+1-1}{w+1} \Rightarrow 1-x=\frac{w}{w+1}$.

So the integral becomes

$$\int_\infty^1 e^{\frac{i\pi\alpha^2}{4(w+1)}} \left(\frac{1}{w+1}\right)^{(\frac{q}{2}-\frac{it}{2}-1)} \left(\frac{w}{w+1}\right)^{(\frac{q}{2}+\frac{it}{2}-1)} \frac{-dw}{(w+1)^2}$$

$$= \int_1^{\infty} \frac{e^{\frac{i\pi\alpha^2}{4(w+1)}} \exp[\frac{it}{2}\log(w+1)]\exp[\frac{it}{2}\log(\frac{w}{w+1})]}{w^{1-\frac{q}{2}}(w+1)^q} dw$$

$$= \int_1^{\infty} \frac{\exp[\frac{i\pi\alpha^2}{4(w+1)} + \frac{it}{2}\log(w)]}{w^{1-\frac{q}{2}}(w+1)^q} dw. \qquad (22)$$

Now the denominator behaves like $w^{q+1-\frac{q}{2}} = w^{1+\frac{q}{2}}$ as $w \to \infty$.

Since the numerator is simply a combination of cosines and sines with modulus equal to 1, the integral converges for all $q > 0$. Now how can we evaluate it? This is where we can use the power of the saddle point method. The phase of the numerator

$$if(w) = i\left[\frac{\pi\alpha^2}{4(w+1)} + \frac{t}{2}\log(w)\right]$$

has two large parameters. If we can find the saddle point, we should be able to arrange for the integration path to pass through the saddle point in such a way and the integral can be estimated using an asymptotic series with terms

$$O\left[\left(\frac{1}{Max(\alpha,t)}\right)^{\frac{n}{2}}\right] \qquad n = 1,2,3,\ldots$$

This will converge very rapidly, since α and t are large. Let us consider $f(w)$.

$$f'(w) = \frac{-\pi\alpha^2}{4(w+1)^2} + \frac{t}{2w} = 0$$

$$\Rightarrow \frac{t(w+1)^2}{2} = \frac{\pi\alpha^2 w}{4} \qquad (23)$$

$$\Rightarrow w^2 + 2w + 1 = \frac{\pi\alpha^2 w}{2}$$

$$\Rightarrow w^2 + (2 - \frac{\pi\alpha^2}{2t})w + 1 = 0,$$

$$\Rightarrow w = \frac{-(2-\frac{\pi\alpha^2}{2t}) \pm \sqrt{4+\frac{\pi^2\alpha^4}{4t^2}-\frac{2\pi\alpha^2}{t}-4}}{2}$$

$$= \frac{\pi\alpha^2}{4t} - 1 \pm \frac{\pi\alpha^2}{4t}\sqrt{1-\frac{8t}{\pi\alpha^2}}$$

$$= \frac{2\alpha^2}{a^2} - 1 \pm \frac{2\alpha^2}{a^2}\sqrt{1-\frac{a^2}{\alpha^2}}$$

$$\Rightarrow w = \frac{2\alpha^2}{a^2}\left[1 \pm \sqrt{1-\frac{a^2}{\alpha^2}}\right] - 1$$

$$\text{where:} \qquad a = \sqrt{\frac{8t}{\pi}} \Rightarrow a^2 = \frac{8t}{\pi}.$$

Here, the positive square root is only interested. So if $\alpha^2 > a^2$, a real saddle point will be at

$$w_{sad} = \frac{2\alpha^2}{a^2}\left[1 + \sqrt{1-\frac{a^2}{\alpha^2}}\right] - 1 \in [1,\infty).$$

If $\alpha^2 < a^2$, then no real saddle point exists since the square root is complex. So a crucial change in behaviour occurs if $\alpha < a = \sqrt{\frac{8t}{\pi}}$ and $\alpha > a$. Denote w_{sad} by the variable pc or $pc(\alpha)$ and $pc(\alpha > a) \in [1,\infty)$. Differentiating again gives

$$f''(w) = \frac{\pi\alpha^2}{2(w+1)^3} - \frac{t}{2w^2}$$

$$f''(w=pc) = \frac{1}{2}\left[\frac{\pi\alpha^2}{(pc+1)^3} - \frac{t}{pc^2}\right]$$

$$= \frac{t}{2pc^2}\left[\frac{\pi\alpha^2}{t(pc+1)^3} - 1\right]$$

from the equation 23 $\Rightarrow \frac{\pi\alpha^2}{t} = \frac{2(pc+1)^2}{pc}$

$$\therefore f''(w=pc)=\frac{t}{2pc^2}\left[\frac{2pc^2(pc+1)^2}{pc(pc+1)^3}-1\right]$$

$$=\frac{t(pc-1)}{2pc^2(pc+1)}>0,$$

so near the saddle point at $w=pc$

$$f(w)\approx f(pc)+\frac{t(pc-1)}{4pc^2(pc+1)}(w-pc)^2+O((w-pc)^3).$$

So let us consider the integral as contour integral of the form

$$\int_{C_{(z)}}\frac{\exp[i(\frac{\pi\alpha^2}{4(z+1)}+\frac{t}{2}\log(z))]}{z^{1-\frac{q}{2}}(z+1)^q}dz.$$

Suppose we choose as our contour the following in the figure 3.1.

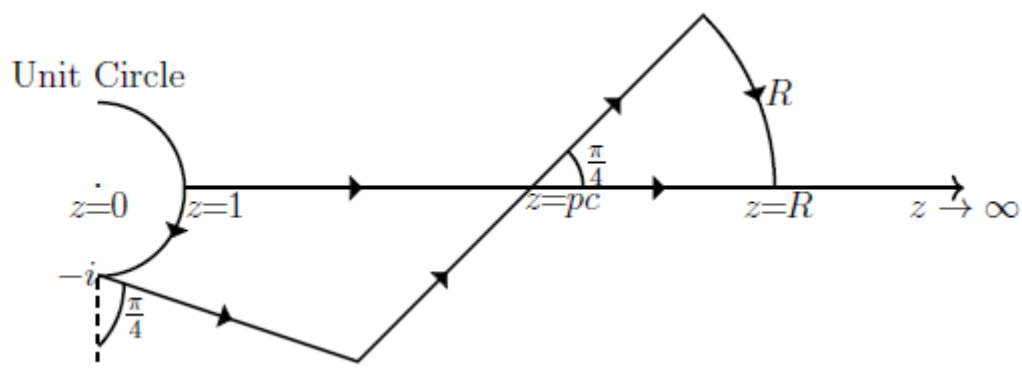

Figure 3.1: Contour of integration in 25.

From the point $z=1$, move along the unit circle to $z=-i$. Then draw a line from $z=-i$ at an angle $\frac{\pi}{4}$ $(z=-i+se^{-i\frac{\pi}{4}}, s>0)$ until it crosses the line originating from $z=pc$ at an angle

$\frac{\pi}{4}$ $(z = pc + ue^{i\frac{\pi}{4}}\ u \in (-r, r))$ as shown. Carry on along this ray until it hits the circle centred at $z = 0$ of radius R $(z = Re^{i\theta}\ \theta \geq 0)$. Move along this circle until you reach $z = R$. In the limit $R \to \infty$ this is equivalent to integrating along the real axis from $z = 1$ to $z = \infty$. Since integrand 22 has no poles for $w \geq 1$.

$$\int_1^\infty - \int_{C_1(0)} - \int_{(-i+se^{-i\frac{\pi}{4}})} - \int_{(pc+ue^{i\frac{\pi}{4}})} - \int_{C_R(0)} = 0 \qquad (24)$$

by Cauchy's Theorem. Hence the integral 22 is given by

$$\int_1^\infty = \int_{C_1(0)} + \int_{(-i+se^{-i\frac{\pi}{4}})} + \int_{(pc+ue^{i\frac{\pi}{4}})} + \int_{C_R(0)} \qquad (25)$$

It would expect that it is the integral through the saddle point that dominates the result. Now consider what the integration $\int_{(pc+ue^{i\frac{\pi}{4}})}$ will equal, where

$$\int_{(pc+ue^{i\frac{\pi}{4}})} = \int_{(pc+ue^{i\frac{\pi}{4}})} \frac{\exp[if(w)]}{w^{(1-\frac{q}{2})}(w+1)^q} dw \qquad (26)$$

Suppose $w = pc + ue^{i\frac{\pi}{4}}; u \in [-r, r] \Rightarrow dw = e^{i\frac{\pi}{4}} du$

i.e near $pc \Rightarrow f(w) = f(pc) + \frac{t(pc-1)}{4pc^2(pc+1)}(w - pc)^2$

$$\Rightarrow f(u) = f(pc) + \frac{iu^2 t(pc-1)}{4pc^2(pc+1)} + O(u^3)$$

So the integral 26 becomes:

$$\approx \int_{-r}^{r} \left[\frac{e^{if(pc)} e^{i\frac{\pi}{4}} \exp[\frac{-u^2 t(pc-1)}{4pc^2(pc+1)}]}{[pc + ue^{i\frac{\pi}{4}}]^{(1-\frac{q}{2})}[pc + 1 + ue^{i\frac{\pi}{4}}]^q} \right] du$$

$$\approx \left(\frac{\exp[\frac{i\pi\alpha^2}{4(pc+1)} + \frac{it}{2}\log(pc) + \frac{i\pi}{4}]}{pc^{(1-\frac{q}{2})}(pc+1)^q} \right) \int_{-r}^{r} e^{-u^2 X(pc)} du$$

where: $X(pc)=\dfrac{t(pc-1)}{4pc^2(pc+1)}$.

Now provided $r^2X(pc)\gg 1$, so that $\exp[-r^2X(pc)]\approx 0$.

We can estimate the integral from tables to give:

$$\left(\frac{2pc\exp[i(\frac{\pi\alpha^2}{4(pc+1)}+\frac{t}{2}\log(pc)+\frac{\pi}{4})]}{pc^{1-\frac{q}{2}}(pc+1)^q}\right)\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]+O\left(\left(\frac{1}{X(pc)}\right)^{\frac{3}{2}}\right)$$

where the value $O\left(\left(\frac{1}{X(pc)}\right)^{\frac{3}{2}}\right)\to 0$ as t gets large and large. Hence the integral 26 becomes:

$$\approx 2\left[\frac{\sqrt{pc}}{pc+1}\right]^q\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]\exp[i(\frac{\pi\alpha^2}{4(pc+1)}+\frac{t}{2}\log(pc)+\frac{\pi}{4})]. \qquad (27)$$

Now if the other integrals in the right hand side from the equation 25 are small, i.e:

$$\left|\int_{C_1(0)}\right|,\left|\int_{C_R(0)}\right|\text{and}\left|\int_{(-i+se^{-i\frac{\pi}{4}})}\right|\ll\left|\int_{(pc+ue^{i\frac{\pi}{4}})}\right|$$

Then our integral 19 equals the result 27 plus $e^{(\frac{i\pi\alpha^2}{4})}$ multiplied by complex conjugate of the result 27 also. Thus,

$$\int_0^1 e^{\frac{i\pi\alpha^2x}{4}}x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx=$$

$$2\left[\frac{\sqrt{pc}}{pc+1}\right]^q\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]\left[\exp[i(\frac{\pi\alpha^2}{4(pc+1)}+\frac{t}{2}\log(pc)+\frac{\pi}{4})]\right.$$

$$\left.+e^{\frac{i\pi\alpha^2}{4}}\exp[-i(\frac{\pi\alpha^2}{4(pc+1)}+\frac{t}{2}\log(pc)+\frac{\pi}{4})]\right]$$

$$=4e^{\frac{i\pi\alpha^2}{8}}\left[\frac{\sqrt{pc}}{pc+1}\right]^q\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]\cos\left[\frac{\pi}{4}+\frac{t}{2}\log(pc)-\frac{\pi\alpha^2(pc-1)}{8(pc+1)}\right].$$

Hence, research arrives at the final estimate for confluent hypergeometric function 18

$$\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4}) \approx \frac{\Gamma(q)}{\Gamma(\frac{q}{2}-\frac{it}{2})\Gamma(\frac{q}{2}+\frac{it}{2})} 4e^{\frac{i\pi\alpha^2}{8}}\left[\frac{\sqrt{pc}}{pc+1}\right]^q\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]$$

$$\times\cos\left[\frac{\pi}{4}+\frac{t}{2}\log(pc)-\frac{\pi\alpha^2(pc-1)}{8(pc+1)}\right] \qquad (28)$$

where $pc=\frac{2\alpha^2}{a^2}\left[1+\sqrt{1-\frac{a^2}{\alpha^2}}\right]-1\in(1,\infty)$, $\quad a=\sqrt{\frac{8t}{\pi}}$ and $\alpha > a$,

which is valid as $t\to\infty$. Also, if $\alpha\to\infty \Rightarrow pc\approx\frac{4\alpha^2}{a^2}-1\Rightarrow pc+1\approx\frac{4\alpha^2}{a^2}$.

$$\Rightarrow X(pc)=\frac{t(pc-1)}{4pc^2(pc+1)}\approx\frac{t(\frac{4\alpha^2}{a^2}-2)}{4(\frac{4\alpha^2}{a^2}-1)^2(\frac{4\alpha^2}{a^2})}$$

$$\approx\frac{ta^4(4\alpha^2-2a^2)}{16\alpha^2(4\alpha^2-a^2)^2}\approx\frac{ta^4(4-2\frac{a^2}{\alpha^2})}{16\alpha^4(4-\frac{a^2}{\alpha^2})^2}$$

where $\alpha\to\infty\Rightarrow\frac{a^2}{\alpha^2}\to 0\Rightarrow X(pc)\approx\frac{t^3}{\pi^2\alpha^4}$, so this should be valid provided $\alpha < t^{\frac{3}{4}}$.

In fact, although we shall not show it here the approximation above is valid for all $\alpha \gg 1$ even $\alpha > t^{\frac{3}{4}}$. So we can also take the limit $\alpha\to\infty$.

Now we will consider the integrals which we have ignored in 25. The integrals $\int_{C_R(0)}\to 0$ as $R\to\infty$ and $\int_{(-i+se^{-i\frac{\pi}{4}})}$ exponentially quite small. Therefore, we shall show that the integral $\int_{C_1(0)}$ does not contribute to the hypergeometric function. We have the integral

$$\int_{C_1(0)}\frac{\exp[\frac{i\pi\alpha^2}{4(w+1)}+\frac{it}{2}log(w)]}{w^{(1-\frac{q}{2})}(w+1)^q}dw \qquad (29)$$

where $C_1(0)$ is the unit circle. Let us to suppose $w = C_1(0) = e^{i\phi}; \phi \in [0, -\frac{\pi}{2}]$. So

$$\frac{1}{w+1} = \frac{1}{(\cos(\phi)+1)+i\sin(\phi)}$$

$$= \frac{(\cos(\phi)+1)-i\sin(\phi)}{(\cos(\phi)+1)^2 + i\sin^2(\phi)}$$

$$= \frac{\cos(\phi)+1}{2(\cos(\phi)+1)} - \frac{i\sin(\phi)}{2(\cos(\phi)+1)}$$

$$= \frac{1}{2} - \frac{i\sin(\phi)}{2(\cos(\phi)+1)}$$

So the integral 29 becomes

$$ie^{\frac{i\pi\alpha^2}{8}} \int_0^{-\frac{\pi}{2}} \frac{\exp[\frac{\pi\alpha^2 \sin(\phi)}{8(\cos(\phi)+1)} - \frac{t\phi}{2}]e^{i\phi}}{(e^{i\phi})^{(1-\frac{q}{2})}[1+e^{i\phi}]^q} d\phi$$

Note that when ϕ is small and negative, the exponent of the exponential term behaves like $-\frac{1}{2}([\frac{\pi\alpha^2}{8} - t]|\phi|)$ since $\alpha > a$ the exponential term declines very rapidly as ϕ changes from 0 to $-\frac{\pi}{2}$. Now make one further substitution

$$x = \frac{\sin(\phi)}{\cos(\phi)+1} \in [0, -1], \text{ where } \phi \in [0, -\frac{\pi}{2}]$$

$$\Rightarrow \frac{dx}{d\phi} = \frac{\cos(\phi)[1+\cos(\phi)] + \sin(\phi)\times\sin(\phi)}{(1+\cos(\phi))^2}$$

$$= \frac{1}{1+\cos(\phi)} = \frac{1}{2}(1+x^2)$$

$$\Rightarrow \quad d\phi = \frac{2}{(1+x^2)}dx$$

$$\sin(\phi) = x(1+\cos(\phi)) = \frac{2x}{1+x^2} \qquad \Rightarrow 1+\cos(\phi) = \frac{2}{1+x^2}$$

$$\therefore e^{i\phi} = \frac{1}{1+x^2}(1-x^2+2ix)$$

$$\phi = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$1+e^{i\phi} = \frac{1}{1+x^2}(2+2ix)$$

$$= \frac{2}{\sqrt{1+x^2}} e^{i\tan^{-1}(x)}$$

so $$\frac{e^{i\phi}}{(e^{i\phi})^{(1-\frac{q}{2})}(1+e^{i\phi})^q} d\phi = \frac{(e^{i\phi})^{\frac{q}{2}}}{(1+e^{i\phi})^q} d\phi$$

$$= \frac{\exp\left[\frac{iq}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]}{\left(\frac{2}{1+x^2}\right)^q \exp[iq\tan^{-1}(x)]} \times \frac{2}{1+x^2} dx$$

$$= \frac{2^{(1-q)}}{(1+x^2)^{(1-q)}} \times \frac{\exp\left[\frac{iq}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]}{\exp[iq\tan^{-1}(x)]} dx$$

$$= \frac{2^{(1-q)}}{(1+x^2)^{(1-q)}} dx$$

because $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}(x);\ x \in (-1,1)$. So the integral 29 transforms to:

$$ie^{\frac{i\pi\alpha^2}{8}} \int_0^{-1} \frac{e^{\frac{\pi\alpha^2 x}{8}} \exp[\frac{-t}{2}\arccos\left(\frac{1-x^2}{1+x^2}\right)]}{2^{(q-1)}(1+x^2)^{(1-q)}} dx.$$

Clearly, this integration will equal a real number times $ie^{\frac{i\pi\alpha^2}{8}}$ i.e:

$$ie^{\frac{i\pi\alpha^2}{8}}\int_0^1 \frac{e^{\frac{\pi\alpha^2 x}{8}}\exp[\frac{-t}{2}\arccos\left(\frac{1-x^2}{1+x^2}\right)]}{2^{(q-1)}(1+x^2)^{(1-q)}}dx = ie^{\frac{i\pi\alpha^2}{8}} \times \text{Real Number} \qquad (30)$$

Now in the confluent hypergeometric function, we have found the integral 19 equals same integral for $x \in (0,\frac{1}{2})$ plus $e^{(\frac{i\pi\alpha^2}{4})}$ multiplied by complex conjugate of same integral for $x \in (\frac{1}{2},1)$. However, the contribution around the unit circle $\int_{C_1(0)}$ is given by 29 for the integral $x \in (0,\frac{1}{2})$. So the contribution to the integral 19 from integrals around the unit circle is:

$$ie^{\frac{i\pi\alpha^2}{8}} \times \textit{Real Number} - ie^{\frac{-i\pi\alpha^2}{8}} e^{\frac{i\pi\alpha^2}{4}} \times \text{Same Real Number}$$

$$= ie^{\frac{i\pi\alpha^2}{8}}[\text{Real Number} - \text{Same Real Number}] = 0.$$

So the integral $\int_{C_1(0)}$ in 30 does not contribute the confluent hypergeometric function. Therefore, our confluent hypergeometric function can be estimated very accurately from the saddle point integral through pc along as we will see in example 3.1.1.

3.1.1 Example 1

Compute the hypergeometric function 17 from $t = 500$ to $t = 1000$ where $\alpha = 100$ and $q = 1.5$. *Maple* will be used to find the first solution by the result 28 and for the second solution by Maple's intrinsic function. In the first case, it has used both of the result 28 and *Maple's intrinsic function* to get the time difference between the time for the first solution and the time for the second solution (T). In the second case, the same methods have used but to get the percentage of error (ε). Two graphs are sketched by using these values with the values of t. So, if we run the code **C** in *Maple*, we will get the result as the two graphs below (figures 3.2 and 3.3). The figure 3.2 illustrates dramatically incremental relationship between t and the time T. When t gets large and large, the time difference between the two solutions will increase gradually, which will be more clearly in the next example 3.1.2. While the figure 3.3 shows the inverse relationship between t and the relative error ε. which means if t becomes larger and larger, relative error will fall dramatically. This gives us a very similar result for the computerized result by Maple's intrinsic function but in a very short time.

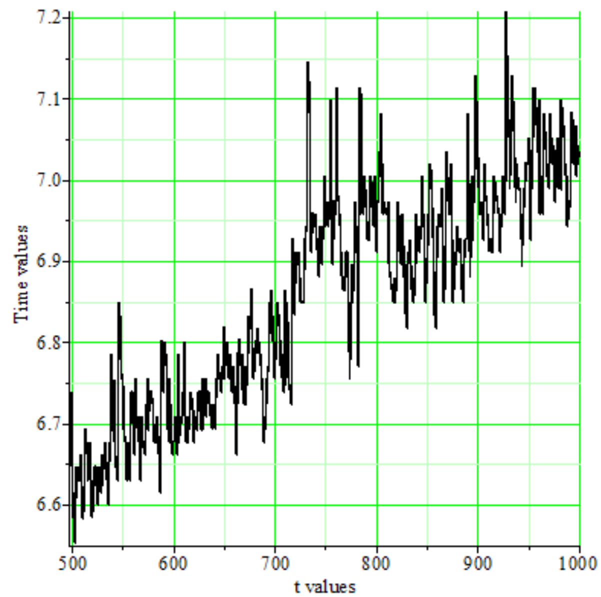

Figure 3.2: Values of $[t]$ With Times $[T]$.

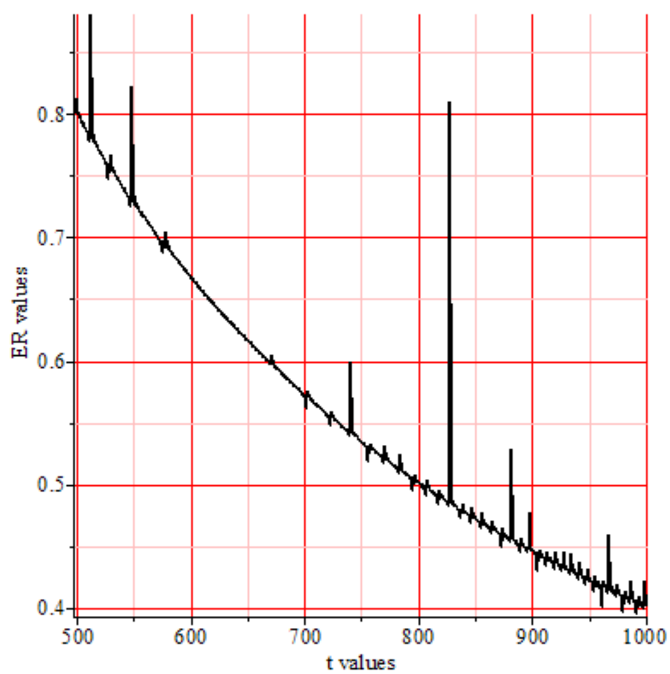

Figure 3.3: Values of $[t]$ with relative error ε.

3.1.2 Example 2

Compute the hypergeometric function 17 if $t = 10000, \alpha = 1000$ and $q = 1.5$. by:

(a) The result 28. (b) Using Maple's intrinsic function.

In the first case(a), if we run the code **D** in *Maple*, we will get the value of the confluent hypergeometric function $= [1.272100242 \times 10^{6816} + 1.288330276 \times 10^{6812} I]$ and $[T = 0.234_s]$ which means it has taken fractions of a second to calculate the confluent hypergeometric function. However, *Maple* had spent for more than 4 hours to get the result using the case (b) and it did not stop.

4 Conclusion

This research has given a brief overview of the hypergeometric function and its importance in the present time. It focused to compute one of commonly used hypergeometric functions in applied mathematics. So, it has found a quick and easy method to compute this function compared with the *Maple* where this method can help to save time for anyone who is interested in these functions. Thus, future research should be to find the most effective methods to simplify the computation of the hypergeometric function, as it is an important factor in the expansion of applied mathematics, which it will open several other domains in applied science.

References

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[2] Carl M. Bender and Steven A. Orszag. "Advanced Mathematical Method for Scientists and Engineers." *Asymptotic Methos and Perturbation Theory.* INC New York: Spring.(1999), chapter 6.

[3] Yudell L.Luke. "The Special functions and their aproximtions." *Academic Press.* New York and London. (1969), chapter 4.

The code of Maple

There are four codes have written for this research. These codes can be received by sending an email to **heshamalzowam@yahoo.com** .

The list of codes:

1. Code **A** computes the series 2 by using the series method.
2. Code **B** computes the hypergeometric function by using Taylor series 4.
3. Code **C** computes the hypergeometric function by using the result 28.
4. Code **D** is a very similar to the code **C** but it uses when t as a single value.