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Approximate solutions for Cauchy-Euler Differential Equations with Riemann-Liouville's Fractional Derivatives via Runge-Kutta Techniques

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الملخص:

في هذا البحث، تمّ مناقشة دراسة الحلول التقريبية للمعادلة كوشي- أويلر التفاضلية مع المشتقات الكسرية من الرتبة $0<\alpha\le 1$، حيث تمّ تطبيق طرق رونج- كوتا على معادلة كوشي-أويلر التفاضلية بمشتقات كسرية بعد تحويلها إلى نظام المعادلات التفاضلية الكسرية؛ وقدمنا أمثلة لتوضيح فعالية هذه الطرق وقارنا النتائج مع الحلول المضبوطة.

Abstract:

The present paper discussed a study on the approximate solutions for the Cauchy-Euler differential equation with fractional derivatives in order $0<\alpha\le 1$. The researchers applied the Runge-Kutta methods to the Fractional Cauchy-Euler differential equation after transforming them into a system of fractional differential equations. The rsearchers further presented examples to illustrate the effectiveness of these methods and compared the results with exact solutions.

Keywords: Cauchy-Euler Fractional differential equations, Riemann -Liouville Fractional derivatives, four-order Runge-Kutta, Runge-Kutta Mersion, fifth-order Runge-Kutta techniques.

1 Introduction

In this article, the researchers study the following Fractional Cauchy-Euler differential equations subject to the conditions:

$$t^{\alpha} D_a^{\alpha} u(t^{\alpha} D_a^{\alpha} u+u)+u=f(t,u(t)),\ t\in[a,b] \tag{1.1}$$

$$u(a)=u_a,\ \ D_a^{\alpha}u(a)=u_a^{(\alpha)},\ \ u(b)=u_b$$

where $0<\alpha\le 1$, $f(x)\in C([a,b]\times\Re;\Re)$ and D_a^{α} are Riemann-Liouville derivatives.

The boundary value problems of ordinary differential equations play an important role in theory and applications and consequence have attracted a great deal of interest over the years. Many authors have studied the fixed-point theorems for the fractional differential equations with the initial conditions, where studied the existence and uniqueness solutions for them. In Bradley et al. [1] introduced the study of a class of linear difference differential equations with multiple advanced arguments where equations are analogous to Cauchy-Euler ordinary differential equations. Cauchy-Euler differential equations often appear in the analysis of computer algorithms, therefore, Delkhosh [3] solved the special type of N-order Cauchy-Euler differential equations by applying a variable change in the equations and after that obtained the conditions, consequently, he obtained an analytical solution for the equations. Ibrahim et al. [8] proposed a new formula for the fractional complex-step method utilizing the Jumarie definition and illustrated an approximate analytic solution for the fractional Cauchy-Euler equations, and applied to image denoising. Pontes [13] presented a study on the solutions of a homogeneous Cauchy-Euler differential equation from the roots of the characteristic equation associated with this differential equation where his solutions were dependent on a polynomial equation of degree n .

In this paper, the researchers have used Runge-Kutta fourth order, Runge-Kutta-Mersion and fifth-order Runge-Kutta methods to solve Cauchy-Euler Fractional differential equation. The rest of the paper is organized as follows: In section 2: fundamental elementary for fractional integral and derivatives differential equations. In section 3: deals with a brief discussion of the Runge-Kutta techniques. In section 4: consists of our selected problem being introduced and the application of Runge- Kutta methods to the selected problem. In section 5: consists of a discussion of illustrative examples and conclusion. Finally, the paper ends with a list of references.

2 Preliminaries

In this section, we introduce some definitions, Fundamental Concepts and properties of the Fractional integral and derivative equations, (see [9,12]).

Definition 1.1. The left and the right Riemann-Liouville fractional integrals $I_{a^+}^{\alpha}$ and $I_{b^-}^{\alpha}$ of order $\alpha > 0$ are defined by

$$I_{a^+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, \; x \in (a,b],$$

and

$$I_{b^-}^{\alpha} f(x) := \frac{1}{\Gamma(n-\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) ds, \; x \in [a,b),$$

where $n-1 < \alpha \le n, n \in \aleph$.

Definition 2.2. The left and the right The Caputo fractional Derivatives ${}^{C}D_{a^+}^{\alpha}$ and ${}^{C}D_{b^-}^{\alpha}$ of order $\alpha > 0$ are defined by

$${}^{C}D_{a^+}^{\alpha} f(x) := (I_{a^+}^{n-\alpha} \circ D^n) f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s) ds, \; x > a,$$

and

$${}^{C}D_{b^-}^{\alpha} f(x) := (-1)^n I_{b^-}^{n-\alpha} \circ D^n f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (s-x)^{n-\alpha-1} f^{(n)}(s) ds, \; x < b,$$

where $n-1 < \alpha \le n, n \in \aleph$.

Definition 2.3. The left and the right Riemann-Liouville fractional Derivatives $D_{a^+}^{\alpha}$ and $D_{b^-}^{\alpha}$ of order $\alpha > 0$ are defined by

$$D_{a^+}^{\alpha} f(x) := D^n \circ I_{a^+}^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-s)^{n-\alpha-1} f(s) ds, \; x > a,$$

and

$$D_{b^-}^{\alpha} f(x) := (-1)^n D^n \circ I_{b^-}^{n-\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (s-x)^{n-\alpha-1} f(s) ds, \; x < b,$$

where $n-1 < \alpha \le n, n \in \aleph$

The basic properties of D_x^{α} are as follows:

Property 2.1. For $R(n)>-1, 0<\alpha\le 1,\ x>0$ and $f(x)\in C[a,b]$, we have

a- $D_{a^+}^{\alpha}(x^n)=\dfrac{\Gamma(1+n)}{\Gamma(1+n-\alpha)}x^{n-\alpha}$

b- ${}^{C}D_{a^+}^{\alpha}(x^n)=\dfrac{\Gamma(1+n)}{\Gamma(1+n-\alpha)}x^{n-\alpha}$

c- $I_x^{\alpha}x^{\gamma}=\dfrac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}s^{\gamma}ds=\dfrac{B(\alpha,1+\gamma)}{\Gamma(1+\gamma+\alpha)}x^{\gamma+\alpha}=\dfrac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}x^{\gamma+\alpha},\quad x>0.$

where we have used $B(\alpha,1+\gamma)=\dfrac{\Gamma(\alpha)\Gamma(1+\gamma)}{\Gamma(\alpha+\gamma)}$.

d- $I_x^{\alpha}I_x^{\beta}f(x)=I_x^{\alpha+\beta}f(x)=I_x^{\beta}I_x^{\alpha}f(x)$, and $D_x^{\alpha}D_x^{\beta}f(x)=D_x^{\alpha+\beta}f(x)=D_x^{\beta}D_x^{\alpha}f(x)$.

Property 2.2 If $f(x)\in AC[a,b]$ and $0<\alpha\le 1$, then:

$$I_{a^+}^{\alpha}\,{}^{C}D_{a^+}^{\alpha}f(x)=f(x)-f(a),$$
$$I_{b^-}^{\alpha}\,{}^{C}D_{b^-}^{\alpha}f(x)=f(x)-f(b).$$

Property 2.3 For $0<\alpha\le 1$ we have:

$$D_{a^+}^{\alpha}f(x)=\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}f(a)+{}^{C}D_{a^+}^{\alpha}f(x)$$

Lemma 2.1. Let $0<\alpha\le 1$ and $f(x)\in C[a,b]$, then $D_x^{\alpha}I_x^{\alpha}f(x)=f(x), x>0$.

Lemma 2. Let $n=[\alpha]+1$ for $\alpha\notin N_0$; $\alpha=n$ for $\alpha\in N_0$, if $f(x)\in AC^n[a,b]$ or $f(x)\in C^n[a,b]$, then $I_a^{\alpha}D_a^{\alpha}f(x)=f(x)-\sum_{k=0}^{n-1}\dfrac{f(a)}{k!}(x-a)^k$.

3 Numerical Methods

The classical 4th-order Runge-Kutta techniques were developed by Runge and Kutta, they introduced the classical formula of the Runge-Kutta method in order four, where this method took a major role in the study of iterative methods on explicit and implicit to apply to solve the ordinary and partial differential equations. In addition, use it to solve systems of ODEs with initial conditions [17,18]. Goeken et al. [6] proposed classical of the Runge-Kutta method with higher derivatives approximate for the 3th and 4th order method. In 1969 England [4] developed another fourth order Runge-Kutta method. The authors constructed the modified Runge-Kutta method and showed that it preserves of accuracy of the original one [10]. Rabiei et al. [14-16] developed the fifth-order improvement Runge-Kutta method for solving ODEs. Hossain et al. [7] presented study on numerical solutions for solving second order initial value problem for ODEs by using Runge-Kutta and Butcher's fifth order Runge-Kutta methods.

Now, consider the initial value problem:

$$y'(x)=f(x,y(x));\quad y(x_0)=y_0 \tag{3.1}$$

Define h to be the time step size and $x_i=x_0+ih$. So, we need some definitions:

Firstly, the formula for the fourth orders Runge-Kutta method for initial value problem (1.1) is given by:

$$k_1 = hf\,(x_i, y_i)$$
$$k_2 = hf\,(x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$
$$k_3 = hf\,(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}) \qquad (3.2)$$
$$k_4 = hf\,(x_i + h, y_i + k_3)$$
$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4); \qquad i = 0,1,2,....$$

Secondly, the improvement version of classical Runge-Kutta method for IVP (1.1) which called Runge-Kutta Mersion (RKM) method with the global error $O(h^4)$, it can be written as the form (see[11]):

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_4 + k_5); \qquad i = 0,1,2,.... \qquad (3.3)$$

where k_1, k_2, k_3, k_4, k_5 are given by:

$$k_1 = hf\,(x_i, y_i)$$
$$k_2 = hf\,(x_i + \frac{h}{3}, y_i + \frac{k_1}{2}) \qquad (3.4)$$
$$k_3 = hf\,(x_i + \frac{h}{3}, y_i + \frac{(k_1 + k_2)}{6})$$
$$k_4 = hf\,(x_i + \frac{h}{2}, y_i + \frac{(k_1 + k_3)}{8})$$
$$k_5 = hf\,(x_i + h, y_i + \frac{1}{2}(k_1 - 3k_3 + 4k_4))$$

with the local truncation error at each step can be using by the following formula :

$$E_r = \frac{1}{3}(2k_1 - 9k_3 + 8k_4 - k_5)$$

Finally, the Butcher's fifth-order Improved Runge-Kutta method (RK5) for IVP (3.1),in this case, the order conditions of the are obtained up to order six and the coefficients of the fifth order method are determined by minimizing the error norm of the sixth order method. The RK–Butcher algorithm of equation (3.1) can be written as the form (see [2,5]):

$$k_1 = hf\,(x_i, y_i)$$
$$k_2 = hf\,(x_i + \frac{h}{4}, y_i + \frac{k_1}{4}) \qquad (3.5)$$
$$k_3 = hf\,(x_i + \frac{h}{4}, y_i + \frac{k_1 + k_2}{8})$$
$$k_4 = hf\,(x_i + \frac{h}{2}, y_i - \frac{k_2}{2} + k_3)$$
$$k_5 = hf\,(x_i + \frac{3h}{4}, y_i + \frac{3k_1 + 9k_4}{16})$$
$$k_6 = hf\,(x_i + h, y_i + \frac{1}{7}(-3k_1 + 2k_2 + 12k_3 - 12k_4 + 8k_5)$$

The fifth-order predictor is defined as:

$$y_{i+1} = y_i + \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6); \qquad i = 0,1,2,.... \qquad (3.6)$$

The errors of the initial value problem (3.1) are calculated by $errors = |y(t_i) - y_i|$, where $y(t_i)$ is the exact solution and y_i is an approximate solution.

4 Main Results

In this section, we applied the pervious numerical methods to find approximate solutions for Cauchy-Euler Fractional differential equation (1.1), we shall transformed the equation (1.1) into a system of fractional differential equations as the follow: let $u = u_1$, so we get:

$$\left.\begin{aligned} D_a^\alpha u_1 &= u_2 \\ D_a^\alpha u_2 &= x^{-2\alpha}(f\,(x, u(x)) - x^\alpha D_a^\alpha u - u) \\ &= x^{-2\alpha} f\,(x, u_1(x)) - x^{-\alpha}\, u_2 - x^{-2\alpha} u_1] \end{aligned}\right\} \qquad (4.1)$$

with initial conditions: $u(a) = u_a, \quad D^\alpha u(a) = u_a^{(\alpha)}$.

Now, we apply the previous techniques, firstly, we compute a numerical solution for system of fractional differential equations (4.1) by using the fourth-order Runge-Kutta method, as follows: we compute:

$$k_{i1} = hf_i\,(x_j, u_{1,j}, u_{2,j})$$
$$k_{i2} = hf_i\,(x_j + \frac{h}{2}, u_{1,j} + \frac{k_{11}}{2}, u_{2,j} + \frac{k_{21}}{2}) \qquad (4.2)$$
$$k_{i3} = hf_i\,(x_j + \frac{h}{2}, u_{1,j} + \frac{k_{12}}{2}, u_{2,j} + \frac{k_{22}}{2})$$
$$k_{i4} = hf_i\,(x_j + h, u_{1,j} + k_{13}, u_{2,j} + k_{23}) \qquad i = 1,2$$

Consequently, we compensate $k_{i1}, k_{i2}, k_{i3}, k_{i4};\ i = 1,2$, in the following iterations:

$$u_{1,j+1} = u_{1,j} + \frac{1}{6}\left(k_{11} + 2k_{12} + 2k_{13} + k_{14}\right); \quad j = 0,1,2,\ldots$$

$$u_{2,j+1} = u_{2,j} + \frac{1}{6}\left(k_{21} + 2k_{22} + 2k_{23} + k_{24}\right) \tag{4.3}$$

Secondly, the Runge-Kutta Mersion method for the Cauchy– Euler fractional differential equations (4.1), can be written in the formula:

$$u_{i,j+1} = u_{i,j} + \frac{1}{6}\left(k_1 + 4k_{i4} + k_{i5}\right); \quad i = 1,2; \quad j = 0,1,2,\ldots \tag{4.4}$$

where: $k_{i1}, k_{i2}, k_{i3}, k_{i4}, k_{i5}$; $i = 1,2$ are taken the formula:

$$\begin{aligned}
k_{i1} &= hf_i(x_j, u_{1,j}, u_{2,j}) \\
k_{i2} &= hf_i(x_j + \frac{h}{3}, u_{1,j} + \frac{k_{11}}{2}, u_{2,j} + \frac{k_{21}}{2}) \\
k_{i3} &= hf_i(x_j + \frac{h}{3}, u_{1,j} + \frac{k_{11}+k_{12}}{6}, u_{2,j} + \frac{k_{21}+k_{22}}{6}) \\
k_{i4} &= hf_i(x_j + \frac{h}{2}, u_{1,j} + \frac{k_{11}+k_{13}}{8}, u_{2,j} + \frac{k_{21}+k_{23}}{8}) \\
k_{i5} &= hf_i(x_j + h, u_{1,j} + \frac{1}{2}\left(k_{11} - 3k_{13} + 4k_{14}\right), u_{2,j} + \frac{1}{2}\left(k_{21} - 3k_{23} + 4k_{24}\right)); \quad i = 1,2.
\end{aligned} \tag{4.5}$$

Finally, the fifth-order Improved Runge-Kutta method (RK5) for Cauchy–Euler fractional differential equations (4.1); can be written as the form (see[2,5,6]):

$$\begin{aligned}
k_{i1} &= hf_i(x_i, u_{1,j}, u_{2,j}) \\
k_{i2} &= hf_i(x_j + \frac{h}{4}, u_{1,j} + \frac{k_{11}}{4}, u_{2,j} + \frac{k_{21}}{4}) \\
k_{i3} &= hf_i(x_j + \frac{h}{4}, u_{1,j} + \frac{k_{11}+k_{12}}{8}, u_{2,j} + \frac{k_{21}+k_{22}}{8}) \\
k_{i4} &= hf_i(x_j + \frac{h}{2}, u_{1,j} - \frac{k_{12}}{2} + k_{13}, u_{2,j} - \frac{k_{22}}{2} + k_{23}) \\
k_{i5} &= hf_i(x_j + \frac{3h}{4}, u_{1,j} + \frac{3k_{11}+9k_{14}}{16}, u_{2,j} + \frac{3k_{21}+9k_{24}}{16}) \\
k_{i6} &= hf_i(x_j + h, u_{1,j} + \bar{\Phi}_1, u_{2,j} + \bar{\Phi}_2) \quad ;
\end{aligned} \tag{4.6}$$

where:

$$\bar{\Phi}_i = \frac{1}{7}(-3k_{i1} + 2k_{i2} + 12k_{i3} - 12k_{i4} + 8k_{i5}); \qquad i = 1,2$$

Consequently; the RK–Butcher of 5th order with six stages of equation (4.1) defined as:

$$u_{i,j+1} = u_j + \frac{1}{90}(7k_{i1} + 32k_{i3} + 12k_{i4} + 32k_{i5} + 7k_{i6}); \quad i = 1,2; \quad j = 0,1,2,\ldots. \tag{4.7}$$

To illustrate the efficiency of these different techniques of solving the Cauchy–Euler fractional differential equations, we compared the numerical results for Runge-Kutta methods with the exact solutions; in the following section.

5 Illustrative Example

In this section, we apply the previous techniques to calculate approximate solutions and absolute errors for the following examples and verify which one is the best and converge to the exact solution quickly or not.

Example 1. Consider the Cauchy–Euler Fractional of nonlinear fractional derivative:

$$x^{\frac{1}{2}} D_a^{\frac{1}{2}} u \left(x^{\frac{1}{2}} D_a^{\frac{1}{2}} u + u \right) + u = 3t^2 + \cos t - x \sin x + \frac{8}{3\sqrt{\pi}} t^2 + \sqrt{x}\cos(x + \tfrac{\pi}{4})$$

with initial conditions: $u(1) = 1.5403,\ D^{\frac{1}{2}} u(1) = 1.2916,\ u(2) = 3.584$ in the closed interval $[1, 2]$, to find approximate solution for Cauchy-Euler fractional differential equation, put $u = u_1$, so we get the following system:

$$D^{\frac{1}{2}} u_1 = u_2$$

$$D^{\frac{1}{2}} u_2 = 3x + \frac{\cos x}{x} - \sin x + \frac{8}{3\sqrt{\pi}} x + \frac{\cos(x + \frac{\pi}{4})}{\sqrt{x}} - \frac{u_2}{\sqrt{x}} - \frac{u_1}{x}$$

with conditions: $u_1(1) = 1.5403,\ \ u_2(1) = 1.2916.$

Table 1. shows the approximating solutions for u of Cauchy-Euler Fractional differential equation, which was obtained by using the Runge-Kutta Butcher methods and their comparison with the exact solution, where the graphical results of the numerical solutions are shown in **Fig.** 1, for step lengths $h = 0.1,\ 0.01,\ 0.001$.

x_i	*Exact* u_1	*RK*	*Error w.r.t. RK*	*RKM*	*Error w.r.t. RKM*	*RK5*	*Error w.r.t. RK5*
1.0	1.5403	1.5403	0.00	1.5403	0.00	1.5403	0.00
1.1	1.6636	1.6755	0.0118994	1.56731	0.096285	1.67554	0.0119437
1.2	1.80236	1.82379	0.0214286	1.59853	0.20383	1.82396	0.0216036
1.3	1.9575	1.98678	0.0292807	1.63469	0.322812	1.98717	0.0296667
1.4	2.12997	2.16617	0.0362044	1.67658	0.453382	2.16684	0.0368764
1.5	2.32074	2.36374	0.0430054	1.72509	0.59565	2.36477	0.044035
1.6	2.5308	2.58135	0.0505507	1.78113	0.749669	2.58281	0.0520073
1.7	2.76116	2.82093	0.0597733	1.84574	0.915419	2.82288	0.061725
1.8	3.0128	3.08448	0.0716774	1.92	1.09279	3.08699	0.0741922
1.9	3.28671	3.37405	0.0873427	2.00512	1.28159	3.3772	0.0904891
2.0	3.58385	3.69178	0.107929	2.10236	1.48149	3.69563	0.111777

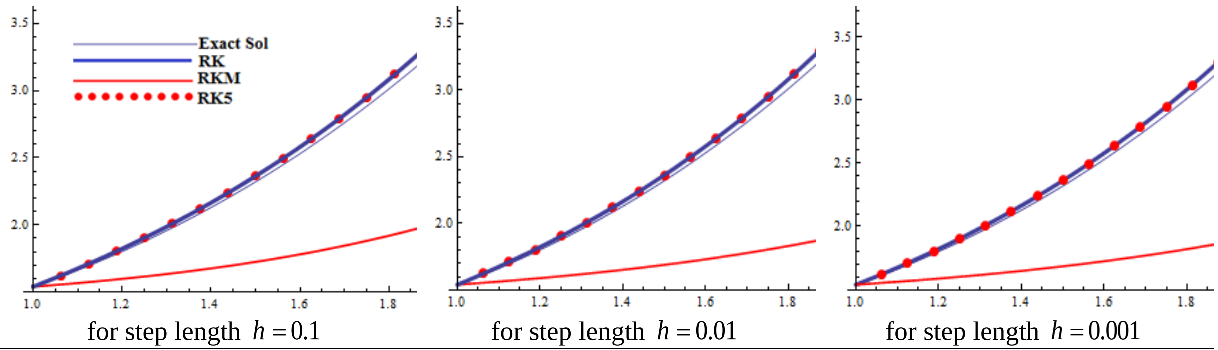


Fig 1. Comparing Approximate solutions of u between Cauchy-Euler Fractional differential equation curve & Runge-Kutta techniques

Example 2. Consider the Cauchy–Euler Fractional of nonlinear fractional derivative:

$$x^{\frac{2}{3}} D^{\frac{2}{3}} u \left(x^{\frac{2}{3}} D^{\frac{2}{3}} u + u \right) + u = \frac{2x^{\frac{3}{2}}}{3\Gamma(\frac{7}{6})} + \frac{2x^{2}}{\Gamma(\frac{5}{3})} + \frac{x^{\frac{3}{2}}}{\Gamma(\frac{11}{6})} + \frac{x^{2}}{\Gamma(\frac{7}{3})} - \frac{8x^{\frac{3}{2}}}{3\sqrt{\pi}} - x^{2}$$

subject to initial conditions: $u(1) = 2.505$, $D^{\frac{2}{3}} u(1) = 3.806$ in the closed interval $[1, 2]$, to find approximate solution for Cauchy-Euler fractional differential equation, put $u = u_1$, so we get:

$$D^{\frac{2}{3}} u_1 = u_2$$

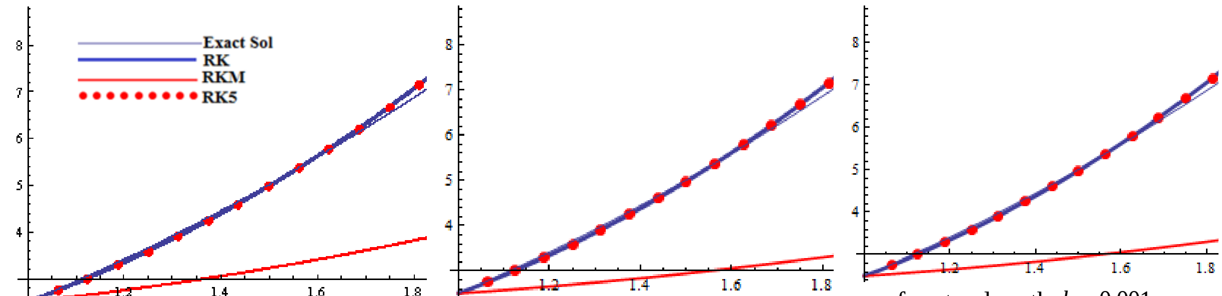
$$D^{\frac{2}{3}} u_2 = \left(\frac{2}{\Gamma(\frac{7}{6})} + \frac{1}{\Gamma(\frac{11}{6})} - \frac{8}{3\sqrt{\pi}} \right) x^{\frac{1}{6}} + \left(\frac{2}{\Gamma(\frac{5}{3})} + \frac{1}{\Gamma(\frac{7}{3})} - 1 \right) x^{\frac{2}{3}} - \frac{u_2}{2x^{\frac{2}{3}}} + \frac{u_1}{x^{\frac{4}{3}}}$$

with respect to conditions: $u_1(1) = 1.505$, $u_2(1) = 3.806$

Table 2 shows the approximating solutions for u of Cauchy-Euler Fractional differential equation, which was obtained by using the Runge-Kutta butcher methods and their comparison with the exact solution, where the graphical results of the numerical solutions are shown in **Fig.** 2, for step lengths $h = 0.1, 0.01, 0.001$.

x_i	Exact u_1	RK	Er w-r-t RK	RKM	Er w-r-t RKM	RK5	Er w-r-t RK5
1.0	1.505	1.505	0.00	1.505	0.00	1.505	0.00
1.1	2.94573	2.90716	0.0385692	2.62324	0.322492	2.907	0.0387334
1.2	3.41772	3.35481	0.0629103	2.75455	0.663171	3.35415	0.063569
1.3	3.92002	3.84882	0.0711969	2.89852	1.0215	3.84734	0.0726765
1.4	4.45222	4.39063	0.0615903	3.05535	1.39686	4.388	0.0642179
1.5	5.01395	4.9817	0.0322529	3.22532	1.78864	4.97759	0.0363587
1.6	5.60491	5.62355	0.0186457	3.40873	2.19617	5.61763	0.0127255
1.7	6.22478	6.3177	0.0929246	3.60597	2.61881	6.30963	0.0848459
1.8	6.87331	7.0657	0.19239	3.81741	3.0559	7.05511	0.181799

1.9	7.55025	7.86909	0.318833	4.04346	3.50679	7.85562	0.305365
2.0	8.25538	8.72942	0.474032	4.28455	3.97084	8.7127	0.457312



for step length $h = 0.1$ for step length $h = 0.01$ for step length $h = 0.001$

Fig 2. Comparing Approximate solutions of u between Cauchy-Euler Fractional differential equation curve & Runge-Kutta techniques

Conclusion

Our main goal is to find the more accurate results in numerical solutions of Cauchy-Euler fractional differential equations by comparing different techniques of Runge-Kutta methods and finding round-off errors. A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size h tends to zero. The accuracy of the solution depends on how small we take the step size h . But it doesn't always, because when we decrease the step length the approximate solutions don't converge rapidly to the exact solution.

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