Solution of Problem of linear Plane Elasticity in Region between an Elliptical Boundary with Cassini Oval Hole by the Boundary Integrals Method

A.S. Deeb
samadeeb0783@gmail.com

Department of Mathematics, Faculty of Science, Elmergib University, Al-Khoms, Libya.

Received: 31/10/2023 | **Accepted:18/11/2023**

الملخص

يتم استخدام طريقة مفكوك فوريبه الحدّي في حل نظام معادلات نظرية المرونة الخطية المستوية للاوساط متجانسة الخواص في الاتجاه، التي تشغل مناطق ثنائية الترابط تحت ضغوط معطاة على الحدود.

ويتم الحصول على الدوال التوافقية الاساسية التي تعطي حل المسألة والازاحات وكذلك الخطأ الناتج من تحقيق الشروط الحدية .

الحالة المدروسة: مجال بيضاوي ذو ثقب كاسيني بيضاوي.

ونحسب أيضا دالة الاجهاد والازاحات داخل المنطقة التي يشغلها الوسط.

الكلمات المفتاحية: نظرية المرونة المستوية، المناطق ثنائية الترابط، الأوساط سوية الخواص في الاتجاه، طريقة التكاملات الحدّية، طريقة التحقيق الحدّى النقطى.

Abstract

A boundary Fourier expansion method is used to solve the system of field equations of plane, linear elasticity in stresses for homogeneous, isotropic media occupying a doubly-connected domain under given pressures on the boundaries. The case understudy is: An elliptic domain with cassini oval. In the case, the boundary values of the relevant harmonic functions are obtained and the error in satisfying the boundary conditions is given. The stress function and the displacement are calculated inside the domain.

Keywords: Plane elasticity; doubly-connected domain; isotropic medium; boundary integral method.

Introduction

The boundary-value problems of plane elasticity for isotropic media have a wide range of applications. They are usually considered as useful approximations to the more realistic three-dimensional problems. When the domain of the solution has complicated geometry, analytical methods become inefficient. The numerical methods stand on the other extreme, but their main disadvantage is that they do not produce formulae for the solution and large computational capabilities are also usually necessary, in addition to the problems raised by the stability of the numerical scheme. In the past few decades, the semi-analytical methods, in combination with the boundary techniques, have gained more popularity as being efficient and require less computational effort than the numerical approaches. Moreover, they produce approximate formulae for the solution and the resulting error can be easily evaluated in many circumstances. Trefftz's method is no doubt the most familiar boundary technique. It requires expansion of the solution in a properly chosen base, then to determine the expansion coefficients using the boundary values of the unknown function (Abou-Dina et al., computational aspects, 2003). Different aspects of this theory related to the completeness property of the used expansion basis and others were considered in(Fairweather et al., 1998), (Tolstov et al., 2012), (Abou-Dina et al., A variant of Trefftz's method, 2004), (Herrera et al., C-complete systems for Stokes problems, 1982). An overview of the method may be found in (Herrera, : a criterion for completeness, 1980). An extensive literature exists on the use of this method, among which we cite (Herrera, Trefftz-Herrera Method, 1997), (Kolodziej et al., Boundary collocation method, 1989), (poullicas et al.). When the basis functions are taken as logarithms of the distance with origins lying outside the domain of solution, this is the Method of Fundamental Solutions treated by many authors (Trefftz et al., 1926), (Kolodziej, Review of application, 1987). An application for doubly-connected regions is carried out in (Kita et al., 1995).

A variant of Trefftz's method, to be used throughout the present work, was suggested by Abou-Dina and Ghaleb (Liu, 2008). It relies on the satisfaction of the boundary conditions, not poinwise, but in the sense of L^2. This method is called the Boundary Fourier Expansion Method (BFEM). It was successfully used to find approximate solutions to several boundary-value problems for Laplace's equation in rectangular domains and others.

In this case, we calculate the boundary values of the two basic harmonic functions through which the solution of the problem is determined in the (BFEM). The error in satisfying the boundary conditions is given. The stress function and the two displacement components are then calculated inside the domain using (BFEM).

Problem formulation

We consider an infinite hollow cylinder of an isotropic elastic medium. Let D be the normal cross-section of the cylinder. This is a two-dimensional, doubly connected region bounded by two contours C_1 and C_2 with parametric representations

$$x_1 = x_1(\theta) \quad \& \quad y_1 = y_1(\theta), \tag{1}$$

$$x_2 = x_2(\theta) \quad \& \quad y_2 = y_2(\theta), \tag{2}$$

where θ is the angular parameter measured, as usual, counter-clockwise from the x-axis of a system of Cartesian coordinates (x, y, z) with center O in the cavity and z -axis along the generators of the cylinder.

The cylinder is acted upon by pressures $p_1(\theta)$ and $p_2(\theta)$ on the lateral surfaces. Thus the considered problem is a generalized Lamé problem.

It is required to find the stresses and the displacement at all points of the cross-section D .

The basic equations and boundary conditions of the two-dimensional theory of elasticity may be found in standard textbooks. Here, we give a brief presentation of these equations along the guidelines given by Abou-Dina and Ghaleb (Zielinski et al.), (Deeb et al., 2018).

Let τ_1 and $\mathbf{n}_1$, τ_2 and $\mathbf{n}_2$ denote respectively the unit vectors tangent and normal to C_1 and C_2 at arbitrary points, the positive sense associated with C_1 and C_2 being taken in the counter-clockwise sense. One has

$$\tau_1 = \frac{\dot{x}_1}{\omega_1} i + \frac{\dot{y}_1}{\omega_1} j \quad \& \quad \mathbf{n}_1 = \frac{\dot{y}_1}{\omega_1} i - \frac{\dot{x}_1}{\omega_1} j, \tag{3}$$

$$\tau_2 = \frac{\dot{x}_2}{\omega_2} i + \frac{\dot{y}_2}{\omega_2} j \quad \& \quad \mathbf{n}_2 = \frac{\dot{y}_2}{\omega_2} i - \frac{\dot{x}_2}{\omega_2} j, \tag{4}$$

where the dot over a symbol denotes differentiation with respect to the parameter θ,

and

$$\omega_1 = \sqrt{\dot{x}_1^{\,2} + \dot{y}_1^{\,2}}, \quad \omega_2 = \sqrt{\dot{x}_2^{\,2} + \dot{y}_2^{\,2}}. \tag{5}$$

In case the contour parameter is the arc length, the corresponding value of ω is unity.

Clearly, the contours C_1 and C_2 should belong, at least, to the class C^1 so as to uniquely define the above defined unit vectors at each point.

Basic equations

In this section, the well-known basic equations governing the plane theory of linear elasticity are presented in accordance with (Zielinski et al.), the representation of harmonic functions is briefly discussed.

Field equations

In the absence of body forces, the stress tensor components in the plane may be expressed by means of one single auxiliary function, called the stress function or Airy's function, subsequently denoted U . In fact, the equations of equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0,$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0. \tag{6}$$

are automatically satisfied if the identically non-vanishing stress components are defined through the function U by the relations:

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \, \partial y}. \tag{7}$$

It is well-known that the biharmonic function may be expressed in terms of two harmonic functions according to the representation

$$U = x\phi + y\phi^c + \psi, \tag{8}$$

where "c" denotes the harmonic conjugate. Thus, the stress components may be rewritten in terms of the harmonic functions as:

$$\sigma_{xx} = x\frac{\partial^2\phi}{\partial y^2} + 2\frac{\partial\phi^2}{\partial y} + y\frac{\partial^2\phi^c}{\partial y^2} + \frac{\partial^2\psi}{\partial y^2},$$

$$\sigma_{xy} = -x\frac{\partial^2\phi}{\partial x\,\partial y} - y\frac{\partial^2\psi^2}{\partial x\,\partial y} - \frac{\partial^2\psi}{\partial x\,\partial y},$$

$$\sigma_{yy} = x\frac{\partial^2\phi}{\partial x^2} + 2\frac{\partial\phi}{\partial x} + y\frac{\partial^2\phi^c}{\partial x^2} + \frac{\partial^2\psi}{\partial x^2} \tag{9}$$

The generalized Hooke's law reads

$$\sigma_{xx} = \frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) + \frac{E}{1+\nu}\frac{\partial u}{\partial x},$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\upsilon}{\partial x}\right),$$

$$\sigma_{yy} = \frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) + \frac{E}{1+\nu}\frac{\partial \upsilon}{\partial y}, \tag{10}$$

where E and ν denote Young's modulus and Poisson's respectively. Using the above relations together with (4), one arrives at:

$$\frac{E}{1+\nu}u = -\frac{\partial U}{\partial x} + 4(1-\nu)\phi,$$

$$\frac{E}{1+\nu}\upsilon = -\frac{\partial U}{\partial y} + 4(1-\nu)\phi^c, \tag{11}$$

which may be rewritten as:

$$2\mu u = (3-4\nu)\phi - x\frac{\partial\phi}{\partial x} - y\frac{\partial\phi^c}{\partial x} - \frac{\partial\psi}{\partial x}, \tag{12}$$

$$2\mu\upsilon = (3-4\nu)\phi^c - x\,\frac{\partial\phi}{\partial y} - y\,\frac{\partial\phi^c}{\partial y} - \frac{\partial\psi}{\partial y}, \tag{13}$$

where $\mu = \dfrac{E}{2(1+\nu)}$ denotes the shear modulus.

$$\begin{aligned}\phi(x,y) &= a_o x + b_o y + c_o xy + d_o(y^2 - x^2)\\ &+\sum_{n=1}^{N}(a_n \cos nx \cosh ny + b_n \cos nx \sinh ny\\ &+c_n \sin nx \cosh ny + d_n \sin nx \sinh ny) + A,\end{aligned} \tag{14}$$

$$\begin{aligned}\phi^c(x,y) &= a_o y - b_o x + \frac{1}{2}c_o(y^2 - x^2) - 2d_o xy\\ &+\sum_{n=1}^{N}(-a_n \sin nx \sinh ny - b_n \sin nx \cosh ny\\ &+c_n \cos nx \sinh ny + d_n \cos nx \cosh ny) + B,\end{aligned} \tag{15}$$

$$\begin{aligned}\psi(x,y) &= f_o x + g_o y + h_o xy + k_o(y^2 - x^2)\\ &+\sum_{n=1}^{N}(f_n \cos nx \cosh ny + g_n \cos nx \sinh ny\\ &+h_n \sin nx \cosh ny + k_n \sin nx \sinh ny) + C.\end{aligned} \tag{16}$$

$$U = x\phi + y\phi^c + \psi \tag{17}$$

$$\begin{aligned}U &= a_o(x^2 + y^2) + \frac{1}{2}c_o(x^2 + y^2) - d_o(x^2 + y^2)x\\ &+\sum_{n=1}^{N} x\,(a_n \cos nx \cosh ny + b_n \cos nx \sinh ny\\ &+c_n \sin nx \cosh ny + d_n \sin nx \sinh ny)\\ &+\sum_{n=1}^{N} y\,(-a_n \sin nx \sinh ny - b_n \sin nx \cosh ny\\ &+c_n \cos nx \sinh ny + d_n \cos nx \cosh ny)\\ &+f_o x + g_o y + h_o xy + k_o(y^2 - x^2)\\ &+\sum_{n=1}^{N}(f_n \cos nx \cosh ny + g_n \cos nx \sinh ny\\ &+h_n \sin nx \cosh ny + k_n \sin nx \sinh ny) + Ax + By + G.\end{aligned} \tag{18}$$

$$\sigma_{nn} = (\sigma_{xx} n_x + \sigma_{xy} n_y)n_x + (\sigma_{xy} n_x + \sigma_{yy} n_y)n_y, \tag{19}$$

$$\sigma_{n\tau} = -(\sigma_{xx} n_x + \sigma_{xy} n_y)n_y + (\sigma_{xy} n_x + \sigma_{yy} n_y)n_x. \tag{20}$$

$$
\begin{aligned}
\sigma_{nn} &= n_x^2(2a_o+3c_o y-2d_o x+2k_o)\\
&+(n_x^2-n_y^2)\Big(\sum_{n=1}^{N} x\,(n^2a_n\cos nx\cosh ny+n^2b_n\cos nx\sinh ny\\
&+n^2c_n\sin nx\cosh ny+n^2d_n\sin nx\sinh ny)\\
&+\sum_{n=1}^{N} y\,(-n^2a_n\sin nx\sinh ny-n^2b_n\sin nx\cosh ny\\
&+n^2c_n\cos nx\sinh ny+n^2d_n\cos nx\cosh ny)\Big)\\
&+(n_x^2+n_y^2)\Big(\sum_{n=1}^{N} 2(-na_n\sin nx\cosh ny-nb_n\sin nx\sinh ny\\
&+nc_n\cos nx\cosh ny+nd_n\cos nx\sinh ny)\Big)\\
&+(n_x^2-n_y^2)\Big(\sum_{n=1}^{N}(n^2f_n\cos nx\cosh ny+n^2g_n\cos nx\sinh ny\\
&+n^2h_n\sin nx\cosh ny+n^2k_n\sin nx\sinh ny)\Big)\\
&+n_y^2(2a_o+c_o y-6d_o x-2k_o)+2n_x n_y(2d_o y-h_o-c_o x)\\
&+2n_x n_y\Big(-\sum_{n=1}^{N} x\,(-n^2a_n\sin nx\sinh ny-n^2b_n\sin nx\cosh ny\\
&+n^2c_n\cos nx\sinh ny+n^2d_n\cos nx\cosh ny)\\
&-\sum_{n=1}^{N} y\,(-n^2a_n\cos nx\cosh ny-n^2b_n\cos nx\sinh ny\\
&-n^2c_n\sin nx\cosh ny-n^2d_n\sin nx\sinh ny)\\
&-\sum_{n=1}^{N}(-n^2f_n\sin nx\sinh ny-n^2g_n\sin nx\cosh ny\\
&+n^2h_n\cos nx\sinh ny+n^2k_n\cos nx\cosh ny)\Big).
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\sigma_{n\tau} &= n_x n_y(-2c_o y-4d_o x-4k_o)\\
&+n_x n_y\Big(\sum_{n=1}^{N} 2x\,(-n^2a_n\cos nx\cosh ny-n^2b_n\cos nx\sinh ny\\
&-n^2c_n\sin nx\cosh ny-n^2d_n\sin nx\sinh ny)\\
&+\sum_{n=1}^{N} 2y\,(n^2a_n\sin nx\sinh ny+n^2b_n\sin nx\cosh ny\\
&-n^2c_n\cos nx\sinh ny-n^2d_n\cos nx\cosh ny)\\
&+\sum_{n=1}^{N} 2(-n^2f_n\cos nx\cosh ny-n^2g_n\cos nx\sinh ny\\
&-n^2h_n\sin nx\cosh ny-n^2k_n\sin nx\sinh ny)\Big)\\
&+(n_x^2-n_y^2)(2d_o y-h_o-c_o x)
\end{aligned}
$$

$$
\begin{aligned}
&+(n_x^2+n_y^2)(-\sum_{n=1}^{N} x\,(-n^2 a_n \sin nx \sinh ny - n^2 b_n \sin nx \cosh ny \\
&+n^2 c_n \cos nx \sinh ny + n^2 d_n \cos nx \cosh ny) \\
&-\sum_{n=1}^{N} y\,(-n^2 a_n \cos nx \cosh ny - n^2 b_n \cos nx \sinh ny \\
&-n^2 c_n \sin nx \cosh ny - n^2 d_n \sin nx \sinh ny) \\
&-\sum_{n=1}^{N}(-n^2 f_n \sin nx \sinh ny - n^2 g_n \sin nx \cosh ny \\
&+n^2 h_n \cos nx \sinh ny + n^2 k_n \cos nx \cosh ny)). \qquad (22)
\end{aligned}
$$

The method of solution:

Short presentation of the method

Let D be a simply-connected region in the plane, bounded by a contour C of finite length L and let $t \in [0,T]$ be a real parameter characterizing the points of the contour C, starting from a point P_0 on C. In particular, t may be the arc length s measured on C anticlockwise as usual, starting from P_0. Extension to doubly-connected domains, the case of present interest, is straightforward.

Consider the following boundary-value problem for the partial differential equation in the unknown function U :

$$K(U(\mathbf{r})) = 0 \quad \text{in} D, \qquad (23)$$

$$W\,U(t) = f(t) \quad \text{on} C, \qquad (24)$$

where $\mathbf{r}$ is the position vector of a general point $P \in D$, K and W are linear partial differential operators and f is a given function on C. Special cases of this problem may be the Dirichlet's, the Neumann's and the mixed boundary-value problems. The case of multiple differential equations and boundary conditions is a straightforward generalization.

Consider now a complete set of linearly independent functions, called the " trial functions" , $\{\varphi_i(\mathbf{r}), i = 0,1,2,...,N\}$. This set of " trial functions" is required to generate the approximate solution $U_a(\mathbf{r})$ as a linear combination of the functions $\varphi_i(\mathbf{r})$ with a certain error tolerance. One such set used for Laplace's equation is the well-known set of Cartesian harmonics

$$\{1, \cos(nx)\cosh(ny), \cos(nx)\sinh(ny), \sin(nx)\cosh(ny), \sin(nx)\sinh(ny), \qquad n = 1,2,...\}$$

in which we are presently interested.

An additional factor determining the choice of the trial functions would be the possibility of satisfaction of some boundary condition on certain parts of the boundary from the outset. Thus, the linear combination

$$U_a(\mathbf{r}) = \sum_{i=0}^{N} a_i\, \varphi_i(\mathbf{r}) \qquad (25)$$

rigorously satisfies equation (4.23) and, possibly, the boundary condition (4.24) on certain parts of the boundary. The number N is usually referred to as the " number of degrees of freedom ". The unknown coefficients $\{a_i, i = 0,1,2...,N\}$ will now be determined so as to enforce the boundary condition on the remaining part of the boundary.

The method proposed hereafter (BFEM) may be considered as a variant of the standard method of approximation of the solution " in the mean" . It generally leads to rectangular systems of linear equations and to integrals that are simpler to evaluate than in the standard method and relies on the following idea: Substitution of (1.3) into (1.2) yields the " error in satisfying the boundary condition" on C :

$$ER(t) \equiv \sum_{n=0}^{N} a_n W \varphi_n(t) - f(t), \qquad t \in [0,T]. \tag{26}$$

Extending the function $ER(t)$ evenly to the interval $[-T,0]$, one obtains a function that, hopefully, should vanish on $[-T,T]$. The Fourier coefficients of this function with respect to the orthonormal set of functions $\{1, \cos\frac{m\pi t}{T}, m = 1,2,...\}$ should then vanish. Setting to zero the first M Fourier coefficients generates a rectangular system of linear algebraic equations of size $M \times N$ for the expansion coefficients $\{a_i, i = 0,1,2...,N\}$ in the form

$$\sum_{n=0}^{N-1} A_{mn} a_n = B_m, \qquad m = 0,1,2,...,M-1, \tag{27}$$

with

$$A_{mn} = \int_0^T W \varphi_n(t) \cos\frac{m\pi t}{T} dt, \quad B_m = \int_0^T f(t) \cos\frac{m\pi t}{T} dt. \tag{28}$$

It may also happen that we do not extend the function $ER(t)$ evenly as explained above, in which case we have to consider all the other Fourier coefficients involving *sines* as well.

The resulting systems of linear algebraic equations will be solved using the well-known method of " Least Squares". The number M may be increased until some error criterion is satisfied. For our purposes, one of two measures of error will be considered hereafter:

1. the maximal boundary error (ERB) measuring the largest error in satisfying the boundary conditions:

$$ERB = \sup_{t \in 0,T]} |ER(t)|, \tag{29}$$

2. the maximal solution error (ERS) measuring the largest error between the approximate solution $U_a(\mathbf{r})$ and the exact solution (assumed known) $U_e(\mathbf{r})$ at a certain properly chosen set of points in the domain of the solution:

$$ERS = \max_k |U_a(\mathbf{r}_k) - U_e(\mathbf{r}_k)|. \tag{30}$$

When the problem under consideration is a Dirichlet's problem, then ERB will be used, since the maximum error in the solution is expected to be reached at the boundary.

For more complicated cases, where there is more than one boundary condition, the same technique may be used invariably. For this, one has only to link additional intervals to $[-T, T]$ corresponding to the additional boundary conditions. This will indeed be the case of the considered problems, when the domain of the solution is doubly-connected and, consequently, there are two boundary conditions to be addressed. Here,

Numerical results

Let the parametric representation of the circular and elliptical normal cross-sections be:

$$x_1(\theta) = a_1 \cos\theta, \qquad y_1(\theta) = a_1 \sin\theta.$$

and the Cassini ovals have Cartesian equation

$$\left(x^2 + y^2 + a^2\right)^2 - 4a^2x^2 = b^4, \qquad b \rangle a.$$

the parametric representation for the normal cross-sections be:

$$x_2 = a_2 \cos\theta \sqrt{\cos 2\theta + \sqrt{\left(\frac{b_2}{a_2}\right)^4 - \sin^2 2\theta}}, \quad y_2 = a_2 \sin\theta \sqrt{\cos 2\theta + \sqrt{\left(\frac{b_2}{a_2}\right)^4 - \sin^2 2\theta}}$$

We take that pressures p_1, p_2 are specified on the two boundaries $C1$, $C2$ in the period $0 < \theta \le 2\pi$.

$$\sigma_{nn} = p_1, \qquad \sigma_{n\tau} = 0.$$

on $C1$,

$$\sigma_{nn} = p_2, \qquad \sigma_{n\tau} = 0.$$

on $C2$.

The above equations are solved numerically using Mathematica software, from which we have acquired the boundary values of the basic harmonic functions ϕ, ϕ^c, ψ, the stress function U and displacements u, v. This is shown on the following figures:

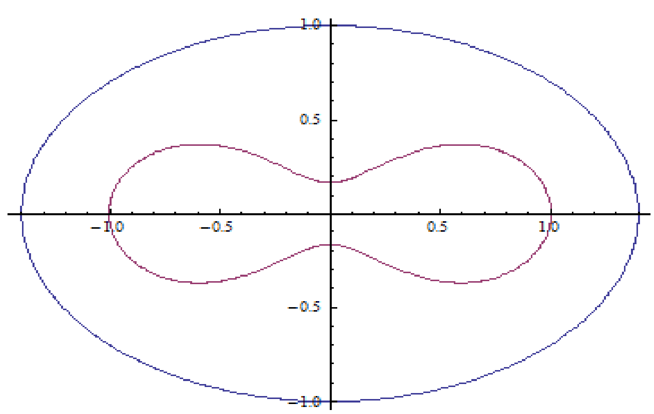
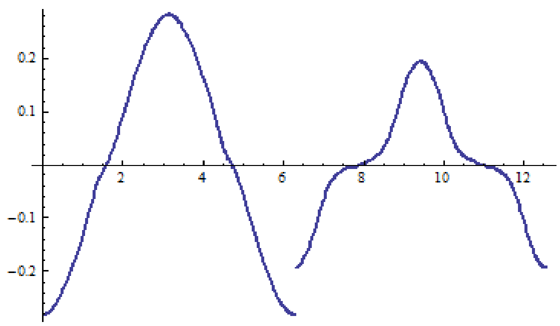


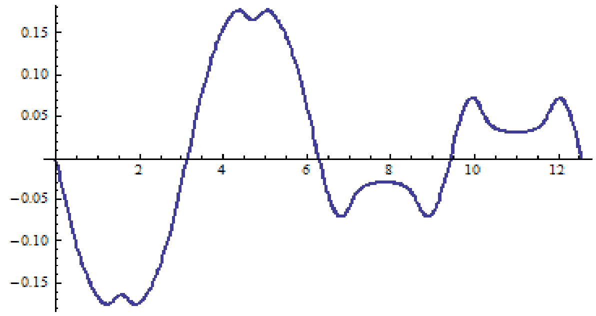
Figure 1: Elliptical normal cross sections with cassini oval hole
$a_1 = 1.4, b_1 = 1, a_2 = 0.7, b_2 = 0.72, p_1 = p_2 = 0.3$

(a) (b)

Φ

Φ^c





Ψ (c)

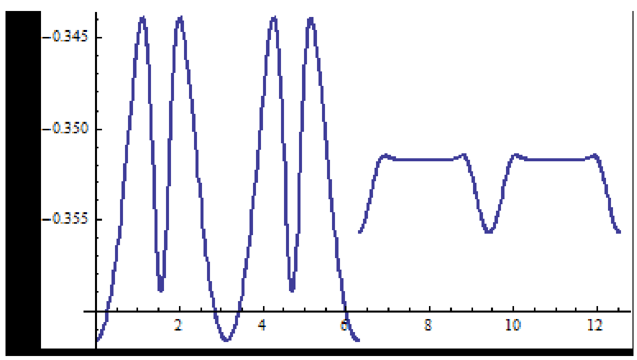


Figure 2: Harmonic function (a) Φ; (b) Φ^c; (c) Ψ on the elliptical cross-section with cassini oval hole.

The error in satisfying the boundary conditions is taken by

$$ERB = \int_0^{2\pi} \left(\overset{(1)}{\sigma}_{nn} + p_1 + \overset{(2)}{\sigma}_{nn} + p_2 \right) d\theta$$

The above equations are solved numerically, their solution provides the boundary values of the basic harmonic functions ϕ, ϕ^c, ψ, the displacements u, v. In BFEM, we used **6** terms in the summations for the different unknown functions, i.e **N = 4** in (27) and (28).

The corresponding number of zeroed Fourier coefficients was **M = 8** for each of the four boundary conditions. The maximum error resulting from the use of this method is $ERB = 0.0038010$.

Further increase of the value of **M** up to **30** kept the results almost unchanged and no instability was observed. We did not go beyond this value of M, but it is thought that there is an upper limit for **M.**

We have also calculated the stress function and the displacements inside the domain from the formulae given above, using the boundary results of BFEM. This is shown on figure (3)

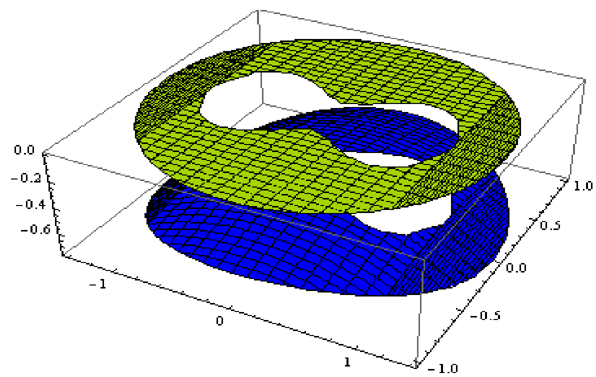
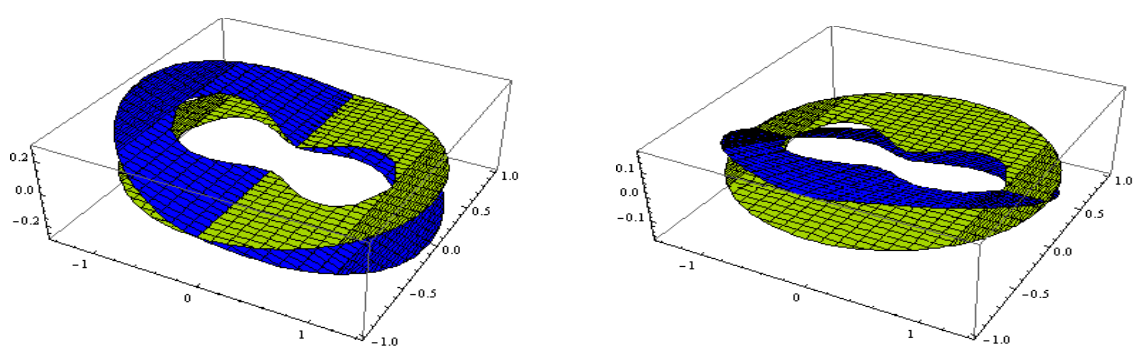


Figure 3: Stress function U in the elliptical domain with cassini oval hole.

It is noticed that the stress function U assumes negative values inside the whole domain. Its surface has the shape of an inverted cup. In absolute values, it becomes larger as one moves away from the internal boundary towards the external one. As for the Cartesian displacement components, they are similar in shape as expected from symmetry.

u v



(a) (b)

Figure 4: Displacements (a) u ; (b) v in the elliptical domain with cassini oval hole.

References

A.S. Deeb, Entesar. Deeb, A. S., Entesar Omar Alarabi, and A. O. El-Refaie. "Solution of some problems of linear plane elasticity in doubly-connected regions by the method of boundary integrals." (2018).

Abou-Dina, M. S., and A. F. Ghaleb. "A variant of Trefftz's method by boundary Fourier expansion for solving regular and singular plane boundary-value problems." *Journal of computational and applied mathematics* 167.2 (2004): 363-387.

Abou-Dina, M. S., and A. F. Ghaleb. "On the boundary integral formulation of the plane theory of elasticity with applications (analytical aspects)." *Journal of computational and applied mathematics* 106.1 (1999): 55-70.

Abou-Dina, M. S., and A. F. Ghaleb. "On the boundary integral formulation of the plane theory of elasticity (computational aspects)." *Journal of computational and applied mathematics* 159.2 (2003): 285-317.

Abou-Dina, Moustafa S. "Implementation of Trefftz method for the solution of some elliptic boundary value problems." *Applied mathematics and computation* 127.1 (2002): 125-147.

Fairweather, Graeme, and Andreas Karageorghis. "The method of fundamental solutions for elliptic boundary value problems." *Advances in Computational Mathematics* 9 (1998): 69-95.

Herrera, Ismael, and Hervé Gourgeon. "Boundary methods, C-complete systems for Stokes problems." *Computer Methods in Applied Mechanics and Engineering* 30.2 (1982): 225-241.

Herrera, Ismael. "Boundary methods: a criterion for completeness." *Proceedings of the National Academy of Sciences* 77.8 (1980): 4395-4398.

Herrera, Ismael. "Trefftz-Herrera Method." *Computer Assisted Mechanics and Engineering Sciences* 4 (1997): 369-382.

Kita, Eisuke, and Norio Kamiya. "Trefftz method: an overview." *Advances in Engineering software* 24.1-3 (1995): 3-12.

Kolodziej, J. A. "Review of application of boundary collocation methods in mechaniccs of continuous media." *SM archives* 12.4 (1987): 187-231.

Kołodziej, J. A., and M. Kleiber. "Boundary collocation method vs FEM for some harmonic 2-D problems." *Computers & structures* 33.1 (1989): 155-168.

Liu, Chein-Shan. "A highly accurate collocation Trefftz method for solving the Laplace equation in the doubly connected domains." *Numerical Methods for Partial Differential Equations: An International Journal* 24.1 (2008): 179-192.

Poullikkas, A., Andreas Karageorghis, and G. Georgiou. "Methods of fundamental solutions for harmonic and biharmonic boundary value problems." *Computational Mechanics* 21.4-5 (1998): 416-423.

Tolstov, Georgi P. *Fourier series*. Courier Corporation, 2012.

Trefftz, Erich. "Ein gegenstuck zum ritzschen verfahren." *Proc. 2nd Int. Cong. Appl. Mech., 1926* (1926).

Zieliński, A. P., and I. Herrera. "Trefftz method: fitting boundary conditions." *International Journal for Numerical Methods in Engineering* 24.5 (1987): 871-891.