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Accelerating the Modified Picard Iteration By Using Green's Function Approach

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Abstract

We consider the Picard's iteration method as a technique for solving initial value problems of the first and second order linear differential equations. The basic idea is the use of Green's function to collect some of the terms in a perfect differential term, and then use the decomposition techniques. For the second order differential equations, we transform the equation to a system of two first order equations and in addition we use the Gauss Seidel technique. The algorithm of the proposed method is discussed. Comparisons with the classical Picard method and modified Picard have illustrated the rapid convergence of the proposed method. Numerical examples have illustrated that the technique obtains the theoretical fixed point quicker than that obtained with other techniques including the modified Picard.

الملخص العربي:

في العقد الأخير من القرن الماضي و إلى الآن لا يزال الاهتمام بنظرية الوجود و الوحدانية مستمراً و ذلك للوصول لحل تقريبي لأي معادلة تفاضلية بشروط ابتدائية، و كانت الطرق التقريبية المتبعة سابقاً متعبة و مملة و تتطلب مجهوداً في العمليات الحسابية.

و لهذا قمت بالبحث عن طريقة أخرى للتخلص من بعض العيوب لتلك الطرق و تعطى نتائج جيدة، فكانت طريقة بيكارد التكرارية و طريقة بيكارد المعدلة هما الأنسب حيث قمت بتعديلها بواسطة معادلة جرين التكاملية و ذلك لزيادة سرعة التقارب والوصول إلى الحل المضبوط لأي مسألة قيمة ابتدائية.

Keywords: Picard iteration, Green's function, Differential equations, Gauss-Seidel method.

1. Introduction

The authors introduced many studies to find solution for differential equation from through using several methods and they said that say the solution must be a unique, among all the available methods, the Picard's method, in [4,7] the authors applied Contraction Mapping Principle which used to prove Picard Theorem "Existence and Uniqueness theorem " which play a vital role in the theory of differential equation, the idea of approach is very simple; the ODE of the first order will be converted to an integral equation, which defines a mapping T, and a conditions of the theorem will imply that T is a contraction which implies that T has a fixed point. There many theorems on the existence of a unique solution of the differential equations under certain condition, and also many approximate techniques for solving systems of ordinary differential equation have been developed.

The Picard iteration is important to construct the existence and uniqueness of solutions of first order of differential equations and apposition to systems of first order of differential equations. In [1], El-Arabawy interested in symbolic computations in treating initial and boundary value problems. In [7] Yilldiz studied nonlinear boundary value problem where they construct an operator equation and show that the special approximations for the operator equation get better convergence speed. Also investigated nonlinear BVP by used the successive approximation method.

The Gauss-Seidel method is a technique for solving systems of linear algebraic equations which was used in researching of Youssef [8] which was concerned with the study of optimizing for Picard iteration method. In [3] Ha studied the Green's function to find numerical solutions of second-order linear and nonlinear differential equations with various boundary conditions. And discussed and analyzed numerical solutions which are obtained by the Green's function and shooting method and comparison between them.

The objective of this paper is to use the Green function for the first order to find solution of initial value problems for first-order systems of ordinary differential equations.

2. Green Functions for First and Second Order Equations

The Green's function is type of function used to solve the non-homogenous differential equation subject to initial and boundary conditions, a homogeneous linear ODE has trivial solution only, but has nontrivial solution when the initial conditions are not zero, a Green's function $G(x,s)$ of linear operator L, at a point s is any solution of the following problem:

$$LG(x,s)=\delta(x-s) \qquad (1)$$

where δ is the Dirac delta function, this method can be solve differential equations of the form;

$$Ly(x)=f(x) \qquad (2)$$

If kernel of L is nontrivial, then the Green's function is not unique. If we multiply the equation (1) for the Green's function by $f(x)$, then by integration we obtain:

$$\int LG(x,s)f(s)ds=\int\delta(x-s)f(s)ds=f(x)$$

Case (I): Consider the first order non-homogeneous equation

$L[y]=f(x)$ for $x>a$ (3)

Where $L=\dfrac{d}{dx}+p(x)$ Subject to initial condition $u(a)=u_0$

The Green function $G(x,s)$ is defined as the solution to $L[G(x,s)]=\delta(x-s)$ subject to $G(a,s)=0$

We can represent the solution to the inhomogeneous problem in Eq. (3) as an integral involving the Green function,

$$y(x) = y_h + \int_a^{\infty} (0) f(s) ds \, (4)$$

Where $G(x,s)$ is continuous in x and s. For $x \neq s$, $LG(x,s) = 0$, the integral also satisfies the initial condition.

$$\int_a^{\infty} G(x,s) f(s) ds = \int_a^{\infty} (0) f(s) ds = 0$$

We integrate the differential equation on the interval (s^-, s^+) to determine this jump.

$$G' + p(x)G = \delta(x-s)$$

$$G(s^+, s) - G(s^-, s) + \int_{s^-}^{s^+} p(x) G(x,s) ds = 1$$

$$G(s^+, s) - G(s^-, s) = 1$$

The homogeneous solution of the differential equation is

$$y_h = y_0 e^{-\int p(\tau) d\tau}$$

Since the Green function satisfies the homogeneous equation for $x \neq s$, it will be a constant times this homogeneous solution for $x < s$ and $x > s$.

$$G(x,s) = \begin{cases} c_1 e^{-\int p(\tau) d\tau}, & a < x < s \\ c_2 e^{-\int p(\tau) d\tau}, & s < x \end{cases}$$

In order to satisfy the homogeneous initial condition $G(a,s) = 0$, the Green function must vanish on the interval (a,s) .

$$G(x,s) = \begin{cases} 0, & a < x < s \\ e^{-\int_s^x p(\tau) d\tau}, & s < x \end{cases}$$

Then the eq. (4) will be come

$$y(x) = y_0 e^{-\int_a^x p(\tau) d\tau} + \int_a^{\infty} e^{-\int_s^x p(\tau) d\tau} f(s) ds \, (5)$$

Consequently, we can rewrite an eq. (5) as the form [2, 9]

$$y(x) = y_0 G(a,x) + \int_a^\infty G(x,s) f(s) ds$$

Where $G(a,x) = e^{-\int_a^x p(\tau)d\tau}$.

Clearly the Green function is of little value in solving the inhomogeneous differential equation in eq. (3), as we can solve that problem directly.

Case (II): We consider the second order non-homogeneous equation

$$L[y] = f(x) \text{ for } \quad a \le x \le b \qquad (6)$$

Subject to a homogeneous boundary conditions, $y(a) = y(b) = 0$.

Where $L = \dfrac{d^2}{dx^2} + p(x)\dfrac{d}{dx} + q(x)$ and where $p(x), q(x)$ and $f(x)$ are continuous functions on interval $[a,b]$.

The Green function $G(x,s)$ is defined as the solution to

$L[G(x,s)] = \delta(x-s)$ subject to $G(a,s) = 0$

The solution of the non-homogeneous problem in eq. (6) as an integral involving the Green function.

$$y(x) = \int_a^b G(x,s) f(s) ds \qquad (7)$$

Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation $L[y] = 0$ Since the Green function satisfies the homogeneous equation for $x \ne s$, it will be a linear combination of the homogeneous solutions.

$$G(x,s) = \begin{cases} c_1 y_1 + c_2 y_2, & x < s \\ d_1 y_1 + d_2 y_2, & x > s \end{cases}$$

Since $G(x,s)$ is continuous and $G'(x,s)$ has onlya jump discontinuity, which determined by:

$$\int_{s^-}^{s^+} [G''(x,s) + p(x)G'(x,s) + q(x)G(x,s)]dx = \int_{s^-}^{s^+} \delta(x-s)dx$$

$$G'(s^+,s) - G'(s^-,s) = 1$$

In [3] studied examples of boundary value problem to find solutions by using Green's function, and in [7] Green's function used in successive approximation equations.

If eq. (6) has initial conditions $y(a) = \alpha_1, y'(a) = \alpha_2$, in this case, the solution of eq. (6) is $y = y_h + y_p$.

where

$$y_p = \int_a^b G(x,s) f(s) ds$$

and

$$y_h'' + p(x)y_h' + q(x)y_h = 0, \qquad y_h(a) = \alpha_1, y_h'(a) = \alpha_2$$

where y_p is solution of the nonhomogeneous equation $Ly_p = f$, which satisfies homogenous boundary conditions $y_p(a) = y_p(b) = 0$ and y_h is solution of the homogenous equation [5,6,9].

3. Materials and Methods

The objective of this work is the use of the Green's function integral approach for the first order equations to accelerate the convergence of Picard iteration method. As well as decompose the system corresponding to the linear second order initial value problems into two parts and use the Green's function integral for one part and use the Gauss seidel approach described in [8].

We consider the second order of linear differential equation (6) with the initial conditions $y(a) = \alpha_1, y'(a) = \alpha_2$,and reducing the eq.(6) to a system of the first order differential equations which takes the form

$$y_1' = y_2 \qquad y_1(a) = \alpha_1$$
$$y_2' = f(x) - p(x)y_2 - q(x)y_1 \qquad , y_2(a) = \alpha_2 \tag{8}$$

Consequently; we use Green's function $G(x,s)$ to find solution for y_2, so it can be write solution of y_2 as the form

$$y_2 = y_2(a)e^{-\int_a^x p(\tau)d\tau} + \int_a^x e^{-\int_s^x p(\tau)d\tau}(f(s) - q(s)y_1)ds$$

where $G(x,s) = e^{-\int_s^x p(\tau)d\tau}$.

Hence, the system of equations (8) can be written as the form

$$y_{1,n} = y_{1,0} + \int_a^x y_{2,n-1}ds \qquad n = 1,2,...$$
$$y_{2,n} = y_{2,0}e^{-\int_a^x p(\tau)d\tau} + \int_a^x e^{-\int_s^x p(\tau)d\tau}(f(s) - q(s)y_{1,n-1})ds \tag{9}$$

According to the [7], we will decrease from the steps of the previous iteration by using the Gauss-Seidel method for linear system subsequently the iteration will become the form

$$y_{1,n} = y_{1,0} + \int_a^x y_{2,n-1}ds, \qquad n = 1,2,...$$
$$y_{2,n} = y_{2,0}e^{-\int_a^x p(\tau)d\tau} + \int_a^x e^{-\int_s^x p(\tau)d\tau}(f(s) - q(s)y_{1,n})ds \tag{10}$$

Since the Picard method for eq. (8) is converges, also this method is converges and will do on increasing of convergence. we will introduced the different cases of differential equations to show the comparison between Picard method and the proposed method.

Firstly, we will consider the linear equation with constant coefficients

$$y'' + \alpha y' + \beta y = f(x) \tag{11}$$

with the initial conditions $y(a)=\alpha_1, y'(a)=\alpha_2$, where the right hand side $f(x)$ of equation takes the different formulas.

Secondly, The Euler Cauchy equations are differential equations of the form

$$\alpha x^2 y''+\beta xy'+\gamma y=f(x) \qquad (12)$$

with given constants α,β,γ and unknown $y(x)$.

4. Main results

In this section, we will mention different types of examples of differential equations, the first type examples are linear with constant coefficients and second type with variable coefficients and the third type are examples of nonlinear differential equations with constant and variable coefficients.

Example 1. Consider the initial value problem

$$y''+4y'+4y=e^{-2x}, \qquad 0\le x\le 1 \qquad y(0)=1,\ y'(0)=-1$$

The nonhomogeneous term is a part of the complementary function of the differential equation, the exact solution is: $y=(1+x+\frac{x^2}{2})e^{-2x}$

The corresponding system takes the form:

$$\begin{aligned} y_1' &= y_2 \qquad\qquad y_1(0)=1 \\ y_2' &= e^{-2x}-4y_2-4y_1 \qquad ,y_2(0)=-1 \end{aligned} \qquad (13)$$

Accordingly, the classical Picard iteration method takes the form

$$y_{1,n} = 1 + \int_0^x y_{2,n-1} ds, \qquad n = 1,2,... \qquad (14)$$

$$y_{2,n} = -1 + \int_0^x (e^{-2x} - 4y_{2,n-1} - 4y_{1,n-1}) ds$$

Hence, the corresponding modified Picard iteration with Green's function integral is

$$y_{1,n} = y_{1,0} + \int_0^x y_{2,n-1} ds, \qquad n = 1,2,... \qquad (15)$$

$$y_{2,n} = y_{2,0} G(x,0) + \int_0^x G(x,s)(e^{-2x} - 4y_{1,n-1}) ds$$

where: $G(x,s) = \begin{cases} 0, & x < s \\ e^{4(s-x)}, & s < x \end{cases} \quad , G(x,0) = e^{-4x}$

Consequently, the corresponding Picard iteration modified by Gauss Seidel with Green's function integral is

$$y_{1,n} = 1 + \int_0^x y_{2,n-1} ds, \qquad n = 1,2,... \qquad (16)$$

$$y_{2,n} = -e^{-4x} + \int_0^x e^{4(s-x)} (e^{-2x} - 4y_{1,n}) ds$$

Then we obtain the solution, and in the following table (1) we give the comparison between the Picard's solution and solution of the system of equations by using Green's function for first order.

x_i	*Exact* y	*Picard* y_7	*Picard with Gr. F* y_7	*Picard G-S. with Gr.* y_7

0.0	1	1	1	1
0.1	0.904697	0.904697	0.904697	0.904697
0.2	0.81779	0.81779	0.81779	0.81779
0.3	0.738152	0.738152	0.738152	0.738152
0.4	0.665007	0.665012	0.665007	0.665007
0.5	0.597804	0.597835	0.597807	0.597804
0.6	0.536126	0.53626	0.536137	0.536126
0.7	0.479631	0.48009	0.479665	0.479631
0.8	0.428021	0.429353	0.428108	0.428021
0.9	0.381014	0.384422	0.381212	0.381014
1.0	0.338338	0.346225	0.338746	0.338338

Table1, the comparison between exact solution and the numerical approximations

Example 2. Consider the initial value problem

$$y''+y'-2y=2x+(x^2-1)e^x, \qquad 0\le x\le 1 \qquad y(0)=0,\ y'(0)=1$$

The nonhomogeneous term contains a part of the complementary function of the differential equation multiplied by a second degree polynomial plus another simple function, the exact solution is:

$$y=-\tfrac{1}{2}-x+\tfrac{88}{81}e^x-\tfrac{95}{162}e^{-2x}+\tfrac{(3x^3-3x^2-7x)}{27}e^x$$

The corresponding system takes the form:

$$\begin{aligned} y_1' &= y_2 \qquad\qquad y_1(0)=0 \\ y_2' &= 2x+(x^2-1)e^x-y_2+2y_1 \qquad ,y_2(0)=1 \end{aligned} \tag{17}$$

Accordingly, the classical Picard iteration method takes the form

$$\begin{aligned} y_{1,n} &= \int_0^x y_{2,n-1}ds, \qquad\qquad n=1,2,\ldots \\ y_{2,n} &= 1+\int_0^x (2s+(s^2-1)e^s-y_{2,n-1}+2y_{1,n-1})ds \end{aligned} \tag{18}$$

Hence, the corresponding modified Picard iteration with Green's function is

$$y_{1,n} = \int_0^x y_{2,n-1} ds, \qquad n = 1,2,... \tag{19}$$

$$y_{2,n} = G(x,0) + \int_0^x G(x,s)(2x + (x^2 - 1)e^x + 2y_{1,n-1}) ds$$

Where: $G(x,s) = \begin{cases} 0, & x < s \\ e^{(s-x)}, & s < x \end{cases}$ $, G(x,0) = e^{-x}$

Consequently, the corresponding Picard iteration modified by Gauss Seidel with Green's function integral is

$$y_{1,n} = \int_0^x y_{2,n-1} ds, \qquad n = 1,2,... \tag{20}$$

$$y_{2,n} = e^{-x} + \int_0^x e^{(s-x)}(2s + (s^2 - 1)e^s + 2y_{1,n}) ds$$

In table (2) we give the comparison between the Picard's solution and the solution by using the Green's function integral of the first order.

x_i	*Exact* y	*Picard* y_7	*Picard with Gr. F* y_7	*Picard G-S. with Gr.* y_7	*Absolute Error*
0.0	0	0.	.0	0.	0.
0.1	0.0908019	0.0908019	0.0908019	0.0908019	0.000000
0.2	0.166192	0.166192	0.166192	0.166192	0.000000
0.3	0.230241	0.230241	0.230241	0.230241	0.000000
0.4	0.286632	0.286634	0.286632	0.286632	0.000000
0.5	0.33885	0.33886	0.338851	0.338849	0.000001
0.6	0.390365	0.390411	0.39037	0.390365	0.000000
0.7	0.444821	0.444975	0.444836	0.44482	0.000001
0.8	0.50623	0.506666	0.506271	0.506226	0.000004
0.9	0.579183	0.580274	0.579282	0.579173	0.000001
1.0	0.669092	0.671566	0.669307	0.669067	0.000025

Table 2, the comparison between exact solution and the numerical approximations

Example 3 Consider the initial value problem

$$y''-3y'+2y=\sin x, \quad 0\le x\le 1 \qquad y(0)=1,\ y'(0)=2$$

The corresponding system of the IVP takes the form

$$\begin{aligned} y_1^{'} &= y_2 && y_1(0)=1 \\ y_2^{'} &= \sin x+3y_2-2y_1 && ,y_2(0)=2 \end{aligned} \qquad (21)$$

Accordingly, the classical Picard iteration method takes the form

$$\begin{aligned} y_{1,n} &= 1+\int_0^x y_{2,n-1}ds, \qquad n=1,2,\ldots \\ y_{2,n} &= 2+\int_0^x (\sin s+3y_{2,n-1}-2y_{1,n-1})ds \end{aligned} \qquad (22)$$

From the system (21), we obtain the Green's function for first order DE y_2, and defined by

$$G(x,s)=\begin{cases} 0, & x<s \\ e^{3(x-s)}, & s<x \end{cases} \quad ,G(x,0)=e^{3x}$$

then the solution to the equationy_2 represented by

$$y_2=2G(x,0)+\int_0^x G(x,s)(\sin s-2y_1)ds \qquad (23)$$

The analogous modified Picard iteration with Green's function integral eq. (21) is approximated as the formula

$$y_{1,n} = 1 + \int_0^x y_{2,n-1} ds, \qquad n = 1,2,\ldots$$
$$y_{2,n} = 2e^{3x} + \int_0^x e^{3(x-s)} (\sin s - 2y_{1,n-1}) ds \qquad (24)$$

And so, the Picard iteration method modified by Gauss Seidel with Green's function integral takes the form

$$y_{1,n} = 1 + \int_0^x y_{2,n-1} ds, \qquad n = 1,2,\ldots$$
$$y_{2,n} = 2e^{3x} + \int_0^x e^{3(x-s)} (\sin s - 2y_{1,n}) ds \qquad (25)$$

We shall display results of the comparison between the Picard iteration as given by the procedure (22), the Picard method modified with Green's function as given by the procedure (24), and the modified Picard by Gauss Seidel method with Green's function as given by the procedure (25), which are indicated in the following tables (3 – 4) on the interval $[0,1]$. Also, the comparisons for all procedures with exact solution are shown in Figure 1.

x_i	*Exact* y	*Picard* $y_{1,7}$	*Picard with Gr. F* $y_{1,7}$	*Picard G-S. with Gr. F.* $y_{1,7}$	*Absolute Error*
0.1	1.22158	1.22158	1.22158	1.22158	0.000000
0.2	1.49338	1.49338	1.49338	1.49338	0.000000
0.3	1.82777	1.82777	1.82777	1.82777	0.000000
0.4	2.24	2.23999	2.24	2.24	0.000000
0.5	2.74879	2.74877	2.7488	2.74879	0.000000
0.6	3.37715	3.37702	3.37718	3.37715	0.000000
0.7	4.15324	4.15279	4.15335	4.15324	0.000000
0.8	5.11162	5.11027	5.112	5.11162	0.000000
0.9	6.29459	6.29103	6.29573	6.29459	0.000000
1.0	7.75396	7.74543	7.75702	7.75396	0.000000

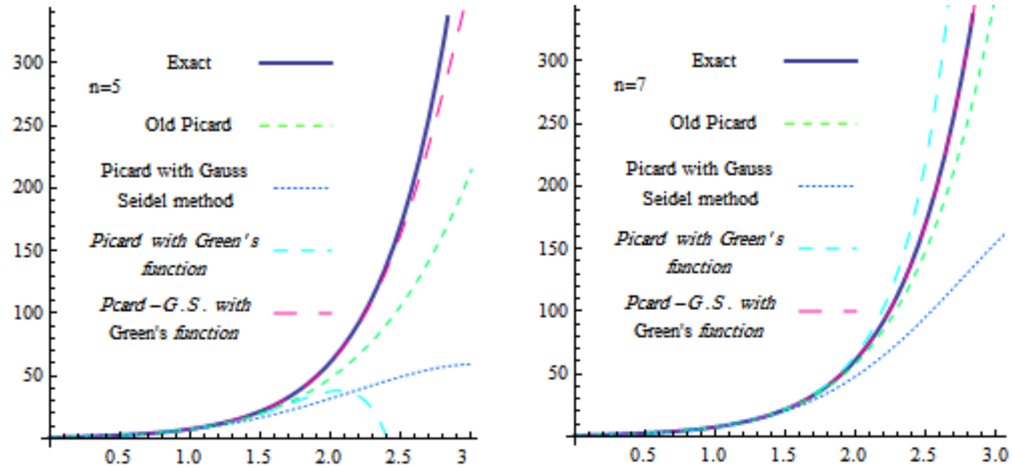
Table 3, the comparison between exact solution and the numerical approximations

From table (3), we find that the procedure (25) give results similar to the exact solution after only seven steps, and the convergence is faster than other procedures to the exact solution.

x_i	*Exact* y	*Picard* $y_{1,12}$	*Picard with Gr. F* $y_{1,12}$	*Picard G-S. with Gr.* $y_{1,12}$
0.1	1.22158	1.22158	1.22158	1.22158
0.2	1.49338	1.49338	1.49338	1.49338
0.3	1.82777	1.82777	1.82777	1.82777
0.4	2.24	2.24	2.24	2.24
0.5	2.74879	2.74879	2.74879	2.74879
0.6	3.37715	3.37715	3.37715	3.37715
0.7	4.15324	4.15324	4.15324	4.15324
0.8	5.11162	5.11162	5.11162	5.11162
0.9	6.29459	6.29459	6.29459	6.29459
1.0	7.75396	7.75396	7.75396	7.75396

Table 4, the comparison between exact solution and
the numerical approximations

Table (4) demonstrates that the comparison between all procedures, where the approximate solutions which we obtained from Picard iteration and the Picard iteration modified with Green's function are arrive to fixed point after five steps from seventh step, while the Picard iteration modified by Gauss Seidel method with Green's function is still save the fixed point after the twelfth step, also will do on increasing of convergence, and is give the accurate results.



(a) Iterates of y_5 (b) Iterates of y_7

Figure 1 Comparison between exact solution and the numerical approximations

Example 4. Consider the Euler equation with initial conditions

$$x^2 y'' + 4xy' + 2y = x^2, \qquad 1 \le x \le 2 \qquad y(1) = 0, \ \ y'(1) = 1$$

The exact solution for the differential eq. will be given $y = \frac{2}{3x} - \frac{3}{4x^2} + \frac{1}{12}x^2$

We rewrite differential equation as the form $y'' + \frac{4}{x}y' + \frac{2}{x^2}y = 1$, and then use the reduction technique to the system of first order differential equations

$$
\begin{aligned}
&{}_1^{'} = y_2 \qquad\qquad y_1(1) = 0 \\
&y_2' = 1 - \tfrac{4}{x}y_2 + \tfrac{2}{x^2}y_1 \qquad , y_2(1) = 1
\end{aligned}
\qquad (26)
$$

So the Green's function is defined by

$$G(x,s)=\begin{cases} 0, & x<s \\ \dfrac{s^4}{x^4}, & s<x \end{cases} \quad ,G(x,1)=\frac{1}{x^4}$$

Therefore, the classical Picard iteration method takes the form

$$\begin{aligned} y_{1,n} &= \int_1^x y_{2,n-1}ds, \qquad n=1,2,... \\ y_{2,n} &= 1+\int_1^x (1-\tfrac{4}{x}y_{2,n-1}-\tfrac{2}{x^2}y_{1,n-1})ds \end{aligned} \qquad (27)$$

Accordingly, the corresponding modified Picard iteration with Green's function integral of (26) takes the form

$$\begin{aligned} y_{1,n} &= \int_1^x y_{2,n-1}ds, \qquad n=1,2,... \\ y_{2,n} &= \frac{1}{x^4}+\int_1^x \frac{s^4}{x^4}(1-\tfrac{2}{x^2}y_{1,n-1})ds \end{aligned} \qquad (28)$$

and the corresponding Picard iteration modified by Gauss Seidel with Green's function takes the form

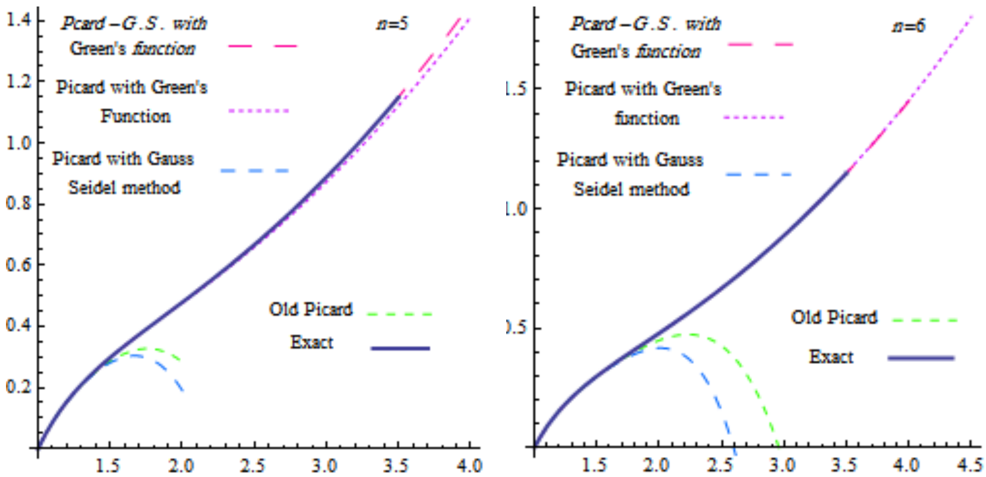
$$\begin{aligned} y_{1,n} &= \int_1^x y_{2,n-1}ds, \qquad n=1,2,... \\ y_{2,n} &= \frac{1}{x^4}+\int_1^x \frac{s^4}{x^4}(1-\tfrac{2}{x^2}y_{1,n})ds \end{aligned} \qquad (29)$$

We introduce the comparison between the Picard iteration, the Picard method modified with Green's function as given by the procedure (28), and the Picard iteration modified by Gauss Seidel with Green's function as given by the procedure (29), which are illustrated in the following tables (5 – 6) on the interval [1,2]. Also, the comparisons for all procedures with exact solution are shown in Figure 2.

x_i	*Exact* y	*Picard* $y_{1,6}$	*Picard with Gr. F.* $y_{1,6}$	*Picard G-S. with Gr. F.* $y_{1,6}$
1.0	0	0	0	0
1.1	0.0870592	0.0870592	0.0870592	0.0870592
1.2	0.154722	0.15472	0.154722	0.154722
1.3	0.209867	0.20984	0.209867	0.209867
1.4	0.256871	0.256714	0.256871	0.256871
1.5	0.298611	0.298009	0.298613	0.298611
1.6	0.337031	0.335276	0.337036	0.337031
1.7	0.373475	0.36923	0.373484	0.373475
1.8	0.408889	0.399917	0.408907	0.408889
1.9	0.443954	0.426825	0.443986	0.443954
2.0	0.479167	0.448953	0.479217	0.479167

Table (5), the comparison between exact solution and
the numerical approximations

From table (5), we find that the procedure (29) give results analogous to the exact solution after only six steps, and the convergence is faster than other procedures to the exact solution.



(a) Iterates of y_5 (b) Iterates of y_6

Figure 2 Comparison between exact solution and the numerical approximations

x_i	*Exact* y	*Picard* $y_{1,14}$	*Picard with Gr. F.* $y_{1,9}$	*Picard G-S. with Gr. F.* $y_{1,6}$
0.0	0	0	0	0
0.1	0.0870592	0.087058	0.0870592	0.0870592
0.2	0.154722	0.154721	0.154722	0.154722
0.3	0.209867	0.209867	0.209867	0.209867
0.4	0.256871	0.256871	0.256871	0.256871
0.5	0.298611	0.29861	0.298611	0.298611
0.6	0.337031	0.33703	0.337031	0.337031
0.7	0.373475	0.373474	0.373475	0.373475
0.8	0.408889	0.408889	0.408889	0.408889
0.9	0.443954	0.443954	0.443954	0.443954
1.0	0.479167	0.479167	0.479167	0.479167

Table 6, the comparison between exact solution and the numerical approximations

Table (6) is summarize the comparison between all procedures, where the approximate solutions which we obtained from Picard iteration and the Picard iteration modified with Green's function are arrive to fixed point after 8 and 3 steps from sixth step, respectively, while the Picard iteration modified by Gauss Seidel method with Green's function is still save the fixed point, also will do on increasing of convergence, and is give the accurate results.

It can be transformed equation $x^2y''+4xy'+2y=x^2$ into differential equations with constant coefficients, and then to find the numerical solutions from the successive approximations, consequently; let $x=e^t \quad \Rightarrow \quad \ln x = t$, so

$$\frac{1}{x}=\frac{dt}{dx}, \quad \frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx} \quad \Rightarrow \quad x\frac{dy}{dx}=\frac{dy}{dt}, \quad x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$$

Thus, $x^2y''+4xy'+2y=x^2$ will become

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=e^{2t}, \quad 0\le t\le 2, \quad y(0)=0, y'(0)=1 \tag{30}$$

Therefore, the exact solution for the differential eq.(30) will be given

$$y(t)=\tfrac{2}{3}e^{-t}-\tfrac{3}{4}e^{-2t}+\tfrac{1}{12}e^{2t}$$

The reducing of the IVP into system of first order takes the form

$$\begin{aligned} y_1' &= y_2 \qquad & y_1(0)&=0 \\ y_2' &= e^{2t}-3y_2-2y_1 \qquad & ,y_2(0)&=1 \end{aligned} \tag{31}$$

Therefor, the classical Picard iteration method takes the form

$$y_{1,n} = \int_0^t y_{2,n-1} d\eta, \qquad n = 1,2,...$$
$$y_{2,n} = 1 + \int_0^t (e^{-2\eta} - 3y_{2,n-1} - 2y_{1,n-1}) d\eta \tag{32}$$

Accordingly, the corresponding modified Picard iteration with Green's function integral of (31) takes the form

$$y_{1,n} = \int_0^t y_{2,n-1} d\eta, \qquad n = 1,2,...$$
$$y_{2,n} = e^{-3t} + \int_0^t e^{3(\eta - t)} (e^{-2\eta} - 2y_{1,n-1}) d\eta \tag{33}$$

and the corresponding Picard iteration modified by Gauss Seidel with Green's function takes the form

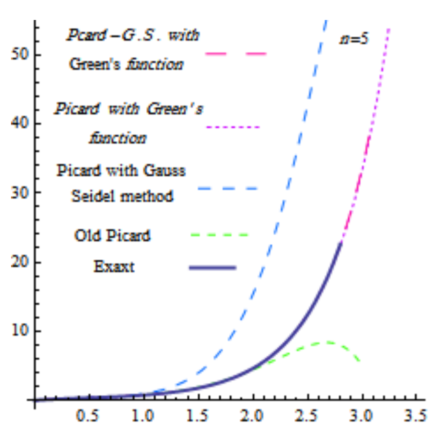
$$y_{1,n} = \int_0^t y_{2,n-1} d\eta, \qquad n = 1,2,...$$
$$y_{2,n} = e^{-3t} + \int_0^t e^{3(\eta - t)} (e^{-2\eta} - 2y_{1,n}) d\eta \tag{34}$$

We introduce the comparison between the Picard iteration as given by the procedure (32), the Picard method modified with Green's function as given by the procedure (33), and the Picard iteration modified by Gauss-Seidel method with Green's function as given by the procedure (34), which are illustrated in the following tables (7 – 8) on the interval [0,2], where the procedure (34) for the proposed method gives us good consequences. Also, the comparisons for all procedures with exact solution are shown in Figure 3.

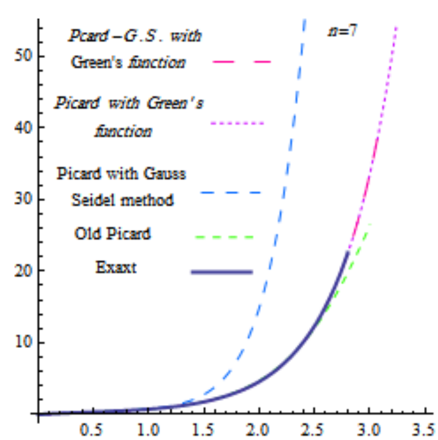
t_i	*Exact* y	*Picard* $y_{1,7}$	*Picard with Gr. F.* $y_{1,7}$	*Picard G-S. with Gr. F.* $y_{1,7}$	*Absolute Error*
0.0	0	0	0	0	0
0.2	0.167399	0.167399	0.167399	0.167399	0.000000
0.4	0.295345	0.295347	0.295345	0.295345	0.000000
0.6	0.416655	0.416695	0.416652	0.416655	0.000000
0.8	0.560883	0.561238	0.560859	0.560883	0.000000
1.0	0.759506	0.761369	0.759396	0.759506	0.000000
1.2	1.05136	1.05828	1.05099	1.05136	0.000000
1.4	1.48918	1.50916	1.48819	1.48918	0.000000
1.6	2.1484	2.19541	2.14619	2.1484	0.000000
1.8	3.13956	3.23075	3.13519	3.13956	0.000000
2.0	4.62633	4.76737	4.61866	4.62633	0.000000

Table 7, the comparison between exact solution and the numerical approximations

From table (7), we estimate that the procedure (34) give results analogous to the exact solution after only seven steps, and very accurate convergence to the exact solution.



(a) Iterates of y_5



(b) Iterates of y_7

Figure 3 Comparison between exact solution and the numerical approximations

t_i	*Exact* y	*Picard* $y_{1,16}$	*Picard with Gr. F.* $y_{1,12}$	*Picard G-S. with Gr. F.* $y_{1,7}$
0.0	0	0	0.	0
0.2	0.167399	0.167399	0.167399	0.167399
0.4	0.295345	0.295345	0.295345	0.295345
0.6	0.416655	0.416655	0.416655	0.416655
0.8	0.560883	0.560883	0.560883	0.560883
1.0	0.759506	0.759506	0.759506	0.759506
1.2	1.05136	1.05136	1.05136	1.05136
1.4	1.48918	1.48918	1.48918	1.48918
1.6	2.1484	2.1484	2.1484	2.1484
1.8	3.13956	3.13956	3.13956	3.13956
2.0	4.62633	4.62632	4.62633	4.62633

Table 8, the comparison between exact solution and the numerical approximations

Table (8) is summarize the comparison between all procedures, where the Picard iteration is not get the fixed point, while the Picard iteration modified with Green's function is arrive to fixed point after 6 steps from seventh step, whilst the Picard iteration modified by Gauss Seidel method with Green's function is still save the fixed point, also will do on decreasing of the steps of the iterations, and is give the accurate results, moreover, gets rapid convergency to the exact solutions.

5. Discussion

The fundamental objective of this work is to find some multipliers that can be used to accelerate the convergence of the Picard iteration method. We find that the ideas of Green's function integration can be used to collect some terms in a single perfect differential term. We used the ideas introduced by Yildiz [7] to decompose the equation, and define Green's function integration to the linear part of the decomposed equation. Also, we considered the Gauss Seidel treatment introduced in Youssef [8]. The new modified Picard iteration method is relatively straightforward to apply at least with the assistance of powerful computer algebra packages and in simple cases it gives exact solutions and in most cases it gives a series that converges rapidly to the unique solution.

The accuracy of the new modified Picard iteration method has been confirmed by comparison with the exact solution as shown in the tables.
Therefore, the method of successive approximations is generally speaking, more widely used: it also used when the expansion of the solution of a differential equation in a power series is impossible, [8]. But this method, unfortunately, has its own shortcoming, which consists in that it calls for the necessity to compute more and more cumbersome integrals.
In a next subsequent work we will try to use this approach to comparison between different numerical methods for solution of differential equations.

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