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Generating countable sets of continuous selfmaps on IN-absorbing spaces.

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Abstract

S(X) is the semigroup, under composition, of all continuous selfmaps of the topological space X. In this paper, Banach's method in [8] is adapted to show that every countable subset of S(X) is contained in a 2- generated subsemigroup of S(X) when X is an IN-absorbing space.

1.Introduction

Let X be an infinite set. In [1] Sierpiński proved the following result:

Theorem 1.1. Every countable family $f_1, f_2, \ldots : X \rightarrow X$ of maps can be generated by two such maps.

In terms of semigroups, Sierpiński proved that any countable subset of the semigroup τ_x, under composition, of all selfmaps on X is contained in a 2- generated subsemigroup of τ_x. A simpler proof was given by Banach in [8].

However, the result of Evans [10], published 17 years later, that any countable semigroup can be embedded in a 2- generated semigroup follows at once from Sierpiński's result. Higman, Neumann and Neumann in [4, Theorem IV] proved that every countable group is embeddable in a 2- generator group. 42 years later, Galvin in[3] proved that every countable set of permutations of X is contained in a 2-generated subgroup of the symmetric group S_X. However, this permutational analogue of

Sierpiński's theorem implies theorem IV in [4]. In [2] Mitchell and Péresse proved that any countable set of surjective maps on an infinite set of cardinality $\aleph_n$ with n ∈ IN can be generated by at most $n^2/2 + 9n/2 + 7$ surjective maps of the same set; and there exist $n^2/2 + 9n/2 + 7$ surjective maps that cannot be generated by any smaller number of surjections. Moreover, in the same paper was presented that several analogous results for other classical transformation semigroups such as the injective maps, Baer – Levi semigroups and the Schützenberger monoids. It is natural to ask if a result, analogous to theorem 1.1, holds when X is endowed with a topological structure.

The symbol S(X) denotes the semigroup, under composition, of all continuous self- maps of the topological space X. It was shown in [9] that any countable subset of S(X) is contained in a 2- generated subsemigroup of S(X) when X is the rationals, the irrationals, the countable discrete space, the cantor space or m-dimensional closed unite cube. The main aim of this paper is, using an elementary technique and different from the one in [9], to prove that a result analogous to theorem 1.1 holds for IN-absorbing spaces.

2. Definitions and theorems

2.1 Theorem[1]**.** Let X be an infinite set. Then any countable subset S of τ_x is contained in a 2- generated subsemigroup of τ_x**.**

Proof[Banach]. Let the countably many members of S be $\theta_1, \theta_2, \ldots$ **.** Partition X into a countable disjoint union of infinitely many sets $X_0, X_1, \ldots, X_n, \ldots$, all of the same cardinality as X, and similarly partition X_0 into $X_{0,1}, X_{0,2}, \ldots, X_{0,n}, \ldots$, again all of the same size as the parent set X.

Let $\beta \in \tau_x$ be any mapping that maps X_n bijectively onto X_{n+1} for all $n \in \mathbb{N} \cup \{0\}$. Our second mapping $\gamma \in \tau_x$ maps X_n bijectively onto $X_{0,n}$ for all $n \geq 1$. Although we have yet to define γ on X_0, we see that mapping $\delta_n = \beta\gamma\beta^n\gamma$ is awell-defined bijection of X onto $X_{0,n}$. We may therefore complete the definition of γ by putting $x\delta_n \gamma = x\theta_n$, ($x \in$ X). Since $\theta_n = \delta_n \gamma$ we obtain the factorization $\theta_n = \beta\gamma\beta^n\gamma^2$ ($n \in \mathbb{N}$).

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2.2 Definition: A topological space is called an IN-absorbing space if it is the disjoint union of countable many infinite subspaces each homeomorphic to X.

2.3 Theorem. Let X be an IN-absorbing space. Then every countable subset of S(X) is contained in a 2- generated subsemigroup of S(X).

Proof. Let X be an IN-absorbing space and $\{f_i\}_{i=1}^{\infty} \subset$S(X). Then by the definition, the space X can be Partitionated into countably many infinite non-empty clopen subsets An $\simeq$ X and n = 0,1,2,3,… . Define ψ: X→X by ψ (x) = $T_n(x)$,

where T_n: $A_n \simeq A_{n+1}$.

It is obvious that ψ is an embedding. Since $A_0 \simeq X$, it can be partitionated into countably many infinite non-empty clopen subsets $B_{0n} \simeq A_0 \simeq X$ and n=1,2,3,… .

We can define a homeomorphism $\propto$: $X \backslash A_0 \rightarrow A_0$ by $\propto (x) = R_n(x)$, where

R_n: $A_n \simeq B_{0n}$ and n=1,2,3,… . Now, define a homeomorphism θ_n: X → B_{0n} by $\propto\psi^n \propto \psi\ (x)$. $\longrightarrow$ (1).

Define a map β : $A_0 \rightarrow X$ by $\beta(x) = f_n\ \theta_n^{-1}(x)$, $x \in B_{0n}$. Thus, β is continuous.

clearly $\beta\ \theta_n(x) = f_n\ (x)$ for $x \in X$ and n =1,2,3,… . $\longrightarrow$ (2)

Define φ: X → X by

$$\varphi(x)=\begin{cases} \propto (x) & ; \ x \in X \backslash A_\circ \\ \beta(x) & ; \ x \in A_\circ \end{cases}$$ .Then

φ is continuous as $\propto$ *and* β are continuous and defined on disjoint clopen subsets

of X. From(1) and (2), we obtain $f_n\ (x)= \varphi^2 \psi^n\ \varphi\psi(x)$ and n = 1,2,3,… .

In other words, $\{f_i\}_{i=1}^{\infty} \subset < \psi, \varphi >$.

2.4 Corollary. Every countable subset of S(P), where P is the irrationals, is contained in 2- generated subsemigroup of S(P).

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Proof. The space $P \simeq \mathrm{IN}^{\mathrm{IN}}$ and $\mathrm{IN}^{\mathrm{IN}}$ is the disjoint union of countable many infinite subspaces -$\mathrm{IN}^{\mathrm{IN}} = \cup\{[n]: n=0,1,2,3,\ldots\}$, so it is an IN-absorbing space .

The following is equivalent to the definition 2.2

2.5 Lemma. A topological space X is an IN-absorbing space if and only if $X \simeq X \times \mathrm{IN}$.

Proof. (⇒)obvious .

(⇐) If $f : X \simeq X \times \mathrm{IN}$, then $f^{-1}(X\times\{i\})$ is non-empty clopen subset of X and it is as a subspace homeomorphic to $X=\cup\{f^{-1}(X\times\{i\}) : i \in \mathrm{IN}\}$.

2.6 Corollary. Every countable subset of S(Q) is contained in a 2-generated subsemigroup of S(Q).

Proof . By theorem (2.3) and lemma (2.5) and the fact that $Q \simeq Q\times \mathrm{IN}$.

2.7 Corollary. Let X be an infinite discrete space. Then every countable subset of S(X) is contained in a 2-generated subsemigroup of S(X).

Proof. By theorem(2.3) and lemma (2.5) and the fact that $X \simeq X \times \mathrm{IN}$.

The space $L = C \setminus\{p\}$ for $p \in C$, where C is the Cantor space, is unique up to homeomorphism [5].

2.8 Corollary. Let $L = C \setminus\{p\}$ for $p \in C$. Then every countable subset of S(L) is contained in a 2-generated subsemigroup of S(L).

Proof. The space L is the unique locally compact, non-compact perfect zero-dimensional space, so homeomorphic to $L \times \mathrm{IN}$ and by theorem (2.3) and lemma (2.5) the proof is completed.

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