ISSN: 2706-9087

ISSUE 8 DESEMBER 2019

مجلة العلوم الإنسانية والتطبيقية
Journal of Humanitarian and Applied Sciences

العدد 8 ديسمبر 2019

Fekete -Szego Problem for Starlike of Complex functions of Order b Related to Generalized Derivative Operator

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الملخص

الغرض من هذه الورقة هو دراسة وحل مشكلة the Fekete-Szego لبعض الفئات الفرعية الجديدة لدالة شبيهة بالنجوم و دالة محدبة مرتبات بالعدد المركب b والتي يرمز لها بالرمز $s_b^{\alpha,n}(m,q,\lambda)$ و $c_b^{\alpha,n}(m,q,\lambda)$ على التوالي التي تم الحصول عليها بعامل تفاضلي معمم $\mathcal{D}^{\alpha,\delta}(m,q,\lambda)$ المعرف في [8] . مختلف الحالات الخاصة المعروفة أو الجديدة لنتائجنا .

Abstract

The purpose of the present paper is to investigate the Fekete-Szego problem for certain new subclasses starlike and convex functions of complex of order b denoted by $s_b^{\alpha,\delta}(m,q,\lambda)$ and $c_b^{\alpha,\delta}(m,q,\lambda)$ respectively, which involving certain a generalized derivative operator $\mathcal{D}^{\alpha,\delta}(m,q,\lambda)$ defined in [8]. Various known or new special cases of our results are also pointed out.

Keywords: Analytic functions; Starlike of complex of order b ; Convex of complex of order b, generalized derivative operator

Introduction and Preliminaries

Fekete-Szegö problem may be considered as one of the most important results about univalent functions, which is related to coefficients of a function's Taylor series and was introduced by Fekete and Szegö [1], studied a special inequality which arises naturally from a combination of two coefficients a_2 and a_3 of a class of univalent analytic normalized function S. They obtained sharp upper bound of $|a_3 - \mu\, a_2^2|$, where μ is real. The problem of maximizing the absolute value of the functional $a_3 - \mu a_2^2$ is called Fekete-Szegö problem. After 30 years or so, Keogh and Merkes [2] solved the problem for certain subclasses of univalent functions. Then Koepf [3] gave excellent results for the class of close-to-convex functions. Moreover, this functional has also been studied for μ as real as well as complex number. And many others follow the

ISSN: 2706-9087

ISSUE 8 DESEMBER 2019

مجلة العلوم الإنسانية والتطبيقية
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العدد 8 ديسمبر 2019

same problems with different techniques for different classes. For other examples defined on various classes can be read in ([3]- [5] and [8]).

Let A denote the family of analytic functions f in the unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ normalized by $f(0)=0=f'(0)-1$. If $f\in A$ then f has the following representation

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k. \qquad (1)$$

We also consider S the class of those functions from A which are univalent in $\mathbb{U}$.
Nasr and Aouf [6, 7] introduced $s^*(b)$ and $C(b)$, the class of starlike functions of complex order and the class of convex functions of complex order respectively. More preciously, the function $f\in A$ is said to be in the class $s^*(b)$, if it satisfies the following condition

$$s^*(b)=\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0,\ z\in\mathbb{U} \text{ and } b\neq 0.$$

Similarly, the function $f\in A$ is said to be in the class $C(b)$, if it satisfies the following condition

$$C(b)=\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0,\ z\in\mathbb{U} \text{ and } b\neq 0.$$

Observe that $s^*(1)$ and $C(1)$ represent standard starlike and convex univalent functions, respectively.

Nagat and Duras in([8],[9]) have recently introduced generalized derivative operator $\mathcal{D}^{\alpha,\delta}(m,q,\lambda)$ as the following:

For the function $f\in A$ given by (1), we define a new generalized derivative operator as follows:

$$\mathcal{D}^{\alpha,\delta}(m,q,\lambda)f(z)=z+\sum_{k=2}^{\infty}k^{\alpha}\left(1+\frac{k-1}{1+q}\lambda\right)^m c(\delta,k)a_k z^k,$$

where $\delta,\alpha\in\mathbb{N}_0=\{0,1,2...\}, m\in\mathbb{Z},\ \lambda,q\geq 0$ and $c(\delta,k)=\frac{(\delta+1)_{k-1}}{(1)_{k-1}}$,

where $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined

by

$$(x)_k = \begin{cases} 1 & for \quad k=0 \\ x(x+1)(x+2)\ldots(x+k-1) & for \quad k \in \mathbb{N} = 1,2,3,\ldots \end{cases}$$

By specializing the parameters of $\mathcal{D}^{\alpha,\delta}(m,q,\lambda)$, we get the following derivative and integral operators.

- The derivative operator introduced by Ruscheweyh [10];

$$\mathcal{D}^{0,n}(0,q,\lambda) \equiv \mathcal{D}^{0,n}(1,0,0);(n \in \mathbb{N}_0) \equiv R^n = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k .$$

- The derivative operator introduced by Sălăgean [12];

$$\mathcal{D}^{\alpha,0}(0,q,\lambda) \equiv \mathcal{D}_1^{0,0}(n,0,1);(n \in \mathbb{N}_0) \equiv D^n = z + \sum_{k=2}^{\infty} k^n a_k z^k .$$

- The generalized Salagean derivative operator introduced by Oboudi [11];

$$\mathcal{D}^{0,0}(n,0,\lambda);(n \in \mathbb{N}_0) \equiv \mathcal{D}_\lambda^n = z + \sum_{k=2}^{\infty} (1+\lambda(k-1))^n a_k z^k .$$

- The generalized Ruscheweyh derivative operator introduced by Darus and Al-Shaqsi [13];

$$\mathcal{D}^{0,n}(1,0,\lambda);(n \in \mathbb{N}_0) \equiv R_\lambda^n = z + \sum_{k=2}^{\infty} (1+\lambda(k-1)) c(n,k) a_k z^k .$$

- The derivative operator introduced by Catas [17];

$$\mathcal{D}^{0,\beta}(m,l,\lambda);(m \in \mathbb{N}_0) \equiv \mathcal{D}^m(\lambda,\beta,l) = z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l} \right)^m c(\beta,k) a_k z^k .$$

- The integral operator introduced by Cho and T. H. Kim [14];

$$\mathcal{D}^{1,0}(-n,\lambda,1) \equiv I_n^\lambda = z + \sum_{k=2}^{\infty} k \left(\frac{1+\lambda}{k+\lambda}\right)^n a_k z^k .$$

Using the operator $\mathcal{D}^{\alpha,\delta}(m,q,\lambda)$, we now introduce the following classes

Definition 1.1. We say that a function $f \in A$ is in the class $s_b^{\alpha,\delta}(m,q,\lambda)$ if

$$\mathrm{Re}\left\{ 1 + \frac{1}{b} \left(\frac{z\,\mathcal{D}^{\alpha,\delta}(m,q,\lambda) f'(z)}{\mathcal{D}^{\alpha,\delta}(m,q,\lambda) f(z)} - 1 \right) \right\} > 0 ,\ z \in \mathbb{U} \text{ and } b \neq 0$$

.

By giving specific values to m, α, δ and b, we obtain the following important subclass studied by $s_{1-b}^{0,n}(0,q,\lambda) = s^*(1-b)$ (Nasr and Aouf [6]).

Definition 1.2. We say that a function $f \in A$ is in the class $C_b^{\alpha,\delta}(m,q,\lambda)$ if

$$C(b) = \text{Re}\left\{1 + \frac{1}{b}\left(\frac{z\,\mathcal{D}^{\alpha,\delta}(m,q,\lambda)f''(z)}{\mathcal{D}^{\alpha,\delta}(m,q,\lambda)f'(z)}\right)\right\} > 0,\ z \in \mathbb{U} \text{ and } b \neq 0$$

Note that $f \in C_b^{\alpha,\delta}(m,q,\lambda) \Leftrightarrow zf' \in S_b^{\alpha,\delta}(m,q,\lambda)$.

2. Main results

Befor we consider how the Taylor series coefficients of functions in the classes $s_b^{\alpha,\delta}(m,q,\lambda)$ and $C_b^{\alpha,\delta}(m,q,\lambda)$ might be bounded, let us first consider this problem for the Caratheodory functions.

Let P be the family of all functions p analytic in $\mathbb{U}$ for which $Re\{p\} > 0$,

given by $p(z) = 1 + c_1 z + c_2 z^2 + \ldots,\quad z \in \mathbb{U}.$

Lemma 2.1([15]) *If $p \in P$ then*

$$|c_k| \leq 2, \qquad \text{for all} \qquad k \in \mathbb{N}.$$

Lemma 2.2([16] *)* If $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with positive real part in $\mathbb{U}$, then

$$|c_2 - \upsilon c_1^2| \leq 2\max\{1, |2\upsilon - 1|\}.$$

The result is sharp for the function

$$p_1(z) = \frac{(1+z)}{(1-z)} \quad or \quad p_1(z) = \frac{(1+z^2)}{(1-z^2)}.$$

Theorem 2.3. Let b be nonzero complex number. If f of the form (1) is in $\mathcal{D}^{\alpha,\delta}(m,q,\lambda)$, then

$$|a_2| \leq \frac{(1+q)|b|}{2^{\alpha-1}(1+q+\lambda)^m(\delta+1)}$$

$$|a_3| \leq \frac{2|b|(1+q)^m}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} max[1, 1+|1+2b|-1].$$

Proof.

By the definition of the class $s_b^{\alpha,\delta}(m,q,\lambda)$ there exists $p \in P$ such, that

$$\frac{z\mathcal{D}^{\alpha,\delta}(m,q,\lambda)f'(z)}{\mathcal{D}^{\alpha,\delta}(m,q,\lambda)f(z)} = 1 - b + b\,p(z),$$

so that,

$$\frac{z\left[1 + 2^{\alpha+1}\left(\frac{1+q+\lambda}{1+q}\right)^m c(\delta,2)a_2 z + 3^{\alpha+1}\left(\frac{1+q+2\lambda}{1+q}\right)^m \frac{(\delta+1)(\delta+2)}{2} a_3 z^2 + \cdots\right]}{z + 2^\alpha\left(\frac{1+q+\lambda}{1+q}\right)^m c(\delta,2)a_2 z^2 + 3^\alpha\left(\frac{1+q+2\lambda}{1+q}\right)^m \frac{(\delta+1)(\delta+2)}{2} a_3 z^3 + \cdots}$$
$$= 1 - b + b[1 + c_1 z + c_2 z^2 + \cdots],$$

which implies the equality

$$z + 2^{\alpha+1}\left(\frac{1+q+\lambda}{1+q}\right)^m c(\delta,2)a_2 z^2 + 3^{\alpha+1}\left(\frac{1+q+2\lambda}{1+q}\right)^m \frac{(\delta+1)(\delta+2)}{2} a_3 z^3 + \cdots$$
$$= z + \left[c_1 b + 2^\alpha\left(\frac{1+q+\lambda}{1+q}\right)^m c(\delta,2)\right] a_2 z^2$$
$$+ \left[c_2 b + c_1 b 2^\alpha\left(\frac{1+q+\lambda}{1+q}\right)^m c(\delta,2) + 3^\alpha\left(\frac{1+q+2\lambda}{1+q}\right)^m \frac{(\delta+1)(\delta+2)}{2}\right] a_3 z^3$$
$$+ \left[c_3 b + \left(\frac{1+q+\lambda}{1+q}\right)^m c(\delta,2)c_2 b + c_1 b 3^\alpha\left(\frac{1+q+2\lambda}{1+q}\right)^m \frac{(\delta+1)(\delta+2)}{2}\right] a_4 z^4 + \cdots$$

Equating the coefficients of both sides we have

$$2^\alpha\left(\frac{1+q+\lambda}{1+q}\right)^m (\delta+1)a_2 = c_1 b\,,$$

$$3^\alpha\left(\frac{1+q+2\lambda}{1+q}\right)^m \frac{(\delta+1)(\delta+2)}{2} a_3 = \frac{b}{2}\left(c_2 - \frac{{c_1}^2}{2}\right) + \frac{(1+2b)}{4}{c_1}^2 b$$

so

ISSN: 2706-9087

ISSUE 8 DESEMBER 2019

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$a_2 = \frac{bc_1(1+q)}{2^{\alpha}(1+q+\lambda)^m(\delta+1)}$, $\quad a_3 = \frac{b(c_2+bc_1^2)(1+q)}{3^{\alpha}(1+q+\lambda)^m(\delta+1)(\delta+2)}$ (2)

Taking into account (2) and Lemma 2.1, we obtain

$$|a_2| = \left|\frac{bc_1(1+q)}{2^{\alpha}(1+q+\lambda)^m(\delta+1)}\right|$$

$$\leq \frac{2|b|(1+q)}{2^{\alpha}(1+q+\lambda)^m(\delta+1)}$$

$$= \frac{|b|(1+q)}{2^{\alpha-1}(1+q+\lambda)^m(\delta+1)}$$

$$|a_3| = \left|\frac{b(1+q)}{3^{\alpha}(1+q+2\lambda)^m(\delta+1)(\delta+2)}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2} + bc_1^2\right]\right|$$

$$\leq \frac{|b|(1+q)}{3^{\alpha}(1+q+2\lambda)^m(\delta+1)(\delta+2)}\left[2 - \frac{|c_1|^2}{2} + \frac{|1+2b|}{2}|c_1|^2\right]$$

$$= \frac{|b|(1+q)}{3^{\alpha}(1+q+2\lambda)^m(\delta+1)(\delta+2)}\left[2 + |c_1|^2\frac{|1+2b|-1}{2}\right]$$

$$= \frac{2|b|(1+q)}{3^{\alpha}(1+q+2\lambda)^m(\delta+1)(\delta+2)} max[1, 1 + |1+2b| - 1].$$

First, we consider the case, when $|a_3 - \mu a_2^2|$ for complex μ.

Theorem 2.4. Let b be a nonzero complex number and let $s_b^{\alpha,\delta}(m,q,\lambda)$.Then for μ complex then

$$|a_3 - \mu a_2^2| \leq \frac{2b(1+q)}{3^{\alpha}(1+q+2\lambda)^m(\delta+1)(\delta+2)} max\left[1, \left|1 + 2b - \frac{2\mu b(1+q)\,(\delta+2)3^{\alpha}(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right|\right]$$

For each _ there is a function in $s_b^{\alpha,\delta}(m,q,\lambda)$ such that equality holds.

Proof. Applying (2) we have

ISSN: 2706-9087

ISSUE 8 DESEMBER 2019

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$$a_3 - \mu a_2^2 = \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)}[c_2 + bc_1^2] - \frac{\mu b^2 c_1^2 (1+q)^2}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)^2}$$

$$= \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)}\left[c_2 + bc_1^2 - \frac{\mu b c_1^2(1+q)\,(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right]$$

$$= \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)}\left[c_2 + \frac{2bc_1^2}{2} - \frac{c_1^2}{2} + \frac{c_1^2}{2} - \frac{2\mu b c_1^2(1+q)\,(\delta+2)3^\alpha(1+q+2\lambda)^m}{2\,2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right]$$

$$= \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + 2b - \frac{2\mu b(1+q)\,(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right)\right], \qquad (3)$$

$$|a_3 - \mu a_2^2| \leq \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)}\left[2 - \frac{|c_1^2|}{2} + \frac{|c_1^2|}{2}\left|1 + 2b - \frac{2\mu b(1+q)\,(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right|\right]$$

$$= \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)}\left[2 + \frac{|c_1^2|^2}{2}\left|1 + 2b - \frac{2\mu b(1+q)\,(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right| - 1\right].$$

Then, with the aid of Lemma 2.2, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{2b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} max\left[1, \left|1 + 2b - \frac{2\mu b(1+q)\,(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right|\right]$$

Taking $\delta = q = m = \alpha = 0$ and b = 1 in Theorem 2.4, we have

Corollary 2.5 [2] If $f \in s^*$, then for $\mu \in C$ we have

$$|a3 - \mu a_2^2| \leq \max\{1, |4\mu - 3|\}.$$

Moreover, for each μ, there is a function in S such that equality holds.

Theorem 2.6 Let $b > 0$, and let $s_b^{\alpha,\delta}(m,q,\lambda)$ .Then for μ real,

Then,

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{2b(1+q)}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m}\left(1+2b\left(1-\mu\dfrac{3^{\alpha}\mu(\delta+2)(1+q+2\lambda)^m(1+q)}{2^{2\alpha}(\delta+1)\ (1+q+\lambda)^{2m}}\right)\right) & if\quad \mu\le\sigma_1,\\ \dfrac{2b(1+q)}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m} & if\ \sigma_1\le\mu\le\sigma_2,\\ \dfrac{2b(1+q)}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m}\left(-1-2b\left(1+\mu\dfrac{3^{\alpha}\mu(\delta+2)(1+q+2\lambda)^m(1+q)}{2^{2\alpha}(\delta+1)\ (1+q+\lambda)^{2m}}\right)\right) & if\quad \mu\ge\sigma_2.\end{cases}$$

Where

$$\sigma_1=\frac{2^{\alpha}(\delta+1)(1+q+\lambda)^{2m}}{3^{\alpha}(1+q)^m(1+q+2\lambda)^m(\delta+2)},$$

$$\sigma_2=\frac{2^{\alpha}(\delta+1)(1+q+\lambda)^{2m}[1+b]}{3^{\alpha}b(1+q)^m(1+q+2\lambda)^m(\delta+2)}.$$

Moreover for each μ, there is a function in $s_b^{\alpha,\delta}(m,q,\lambda)$ such that equality hold

Proof By (3), we obtain

$$a_3-\mu a_2^2=\frac{b(1+q)}{3^{\alpha}(1+q+2\lambda)^m(\delta+1)(\delta+2)}\Bigg[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left(1+2b-\frac{2\mu b(1+q)\ (\delta+2)3^{\alpha}(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right)\Bigg]$$

First, let. $\mu\le\sigma_1$, In this case, by (3), Lemma 2.1 and give

$$|a_3-\mu a_2^2|\le\frac{b(1+q)}{3^{\alpha}(1+q+2\lambda)^m(\delta+1)(\delta+2)}\Bigg[2-\frac{|c_1^2|}{2}+\frac{|c_1^2|}{2}\left|1+2b-\frac{2\mu b(1+q)\ (\delta+2)3^{\alpha}(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right|\Bigg]$$

$$\le\frac{2b(1+q)}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m}\left(1+2b\left(1-\mu\frac{3^{\alpha}\mu(\delta+2)(1+q+2\lambda)^m(1+q)}{2^{2\alpha}(\delta+1)\ (1+q+\lambda)^{2m}}\right)\right)$$

Now let $\sigma_1\le\mu\le\sigma_2$, Then, using the above calculations, we get

$$|a_3-\mu a_2^2|\le \frac{2b(1+q)}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m}$$

Finally, if $\mu \ge \sigma_2$, then we obtain

$$|a_3-\mu a_2^2| \le \frac{b(1+q)}{3^{\alpha}(1+q+2\lambda)^m(\delta+1)(\delta+2)}\left[2-\frac{|c_1^2|}{2}+\frac{|c_1^2|}{2}\left|-1-2b+\frac{2\mu b(1+q)\,(\delta+2)3^{\alpha}(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)}\right|\right]$$

$$\le \frac{2b(1+q)}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m}\left(-1-2b\left(1+\mu\frac{3^{\alpha}\mu(\delta+2)(1+q+2\lambda)^m(1+q)}{2^{2\alpha}(\delta+1)\ (1+q+\lambda)^{2m}}\right)\right).$$

Using the well-known Alexander relation, we easily obtain bounds of coefficients and a solution of the Fekete-Szegö problem for the class $c_b^{\alpha,\delta}(m,q,\lambda)$

Theorem 2.7. Let b be nonzero complex number. If f of the form (1) is in $\mathcal{D}^{\alpha,\delta}(m,q,\lambda)$, then

$$|a_2| \le \frac{(1+q)|b|}{2^{\alpha}(1+q+\lambda)^m(\delta+1)}$$

$$|a_3| \le \frac{2(1+q)|b|}{3^{\alpha+1}(1+q+2\lambda)^m(\delta+1)(\delta+2)}[1+|1+2b|]$$

$$|a_3-\mu a_2^2| \le \frac{2b(1+q)}{3^{\alpha+1}(1+q+2\lambda)^m(\delta+1)(\delta+2)} max\left[1,\left|1+2b-\frac{3\mu b(1+q)\ (\delta+2)3^{\alpha}(1+q+2\lambda)^m}{2^{2\alpha+1}(1+q+\lambda)^{2m}(\delta+1)}\right|\right].$$

Taking $\delta = q = m = \alpha = 0$ and $b = 1$ in Theorem 2.7, we have

Corollary 2.8 [2] *S If $f \in s^*$, then for $\mu \in C$ we have*

$$|a3 - \mu a_2^2| \leq max\left\{\frac{1}{3}, |\mu - 1|\right\}.$$

Acknowledgment. The authors would like to express sincere thanks to the referee for careful reading and suggestions which helped us to improve the paper.

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