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Fekete-Szegö Inequalities for Certain subclasses of p-Valent Functions of Complex Order Associated with Fractional Derivative Operator

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Abstract

In the present paper, we obtain Fekete-Szegö inequalities and sharp bounds for some subclasses of analytic and p-valent functions in the open unit disk defined by certain fractional derivative operator.

Keywords: p-valent function, subordination, starlike function, convex function, fractional derivative operator, Fekete-Szegö inequality.

Introduction And Definitions

Let $A(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \qquad (p \in \mathbb{N}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$.

Let $f(z)$ and $g(z)$ be functioning analytic in $\mathcal{U}$, we say that the function $f(z)$ is a subordinate to $g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathcal{U})$, such that $f(z) = g(w(z))$ for all $z \in \mathcal{U}$.

This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$. It is well known that, if the function $g(z)$ is univalent in $\mathcal{U}$, $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let $\phi(z)$ be an analytic function with $\phi(0) = 1$, $\phi'(0) > 0$ and $\mathrm{Re}(\phi(z)) > 0$ $(z \in \mathcal{U})$, which maps the open unit disk $\mathcal{U}$ onto a region starlike with respect to 1

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and is symmetric with respect to the real axis . Ali et al. [1] defined and studied the class $S^*_{b,p}(\phi)$ to be the class of functions $f(z) \in A(p)$ for which

$$1+\frac{1}{b}\left\{\frac{1}{p}\frac{zf'(z)}{f(z)}-1\right\} \prec \phi(z), \qquad (z \in \mathcal{U}, b \in \mathbb{C}\backslash\{0\}) \tag{1.2}$$

and the class $C_{b,p}(\phi)$ of all functions for which

$$1-\frac{1}{b}+\frac{1}{bp}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \prec \phi(z), \qquad (z \in \mathcal{U}, b \in \mathbb{C}\backslash\{0\}) \tag{1.3}$$

Note that $S^*_{1,1}(\phi) = S^*(\phi)$ and $C_{1,1}(\phi) = C(\phi)$, The classes were introduced and studied by Ma and Minda [2]. The familiar class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order $\alpha,\ 0 \le \alpha < 1$ are the special cases of $S^*_{1,1}(\phi)$ and $C_{1,1}(\phi)$, respectively, when

$$\phi(z) = \frac{1+(1-2\alpha)z}{1-z}.$$

We recall the following definitions of fractional derivative operators which were used by Owa [4] and see [6] and [7] as follows:

Definition 1.1. The fractional derivative operator of order λ is defined, for a function $f(z)$, by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^\lambda}d\xi\,, \qquad 0 \le \lambda < 1 \tag{1.4}$$

where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

With the aid of the above definition, we define a generalization of the fractional derivative operator $\Omega^{\lambda,p}_{0,z}$ by

$$\Omega^{\lambda,p}_{0,z} f(z) = \frac{\Gamma(1+p-\lambda)}{\Gamma(1+p)}\, z^\lambda\, D^\lambda_{0,z} f(z) \tag{1.5}$$

for $f(z) \in A(p)$, $p \in \mathbb{N}$ and $0 \le \lambda < 1$. Then it is observed that $\Omega^{\lambda,p}_{0,z} f(z)$ maps $A(p)$ onto itself as follows:

$$\Omega^{\lambda,p}_{0,z} f(z) = z^p + \sum_{n=1}^{\infty} \varphi_n(\lambda,p)\, a_{p+n}\, z^{p+n} \ , \tag{1.6}$$

where

$$\varphi_n(\lambda,p)=\frac{\Gamma(1+p-\lambda)\Gamma(1+p+n)}{\Gamma(1+p)\Gamma(1+p-\lambda+n)}\ ,\qquad (n\in\mathbb{N}) \tag{1.7}$$

We let $\varphi_n(\lambda,p)\equiv\varphi_n$, and notice that

$$\Omega_{0,z}^{0,p}f(z)=f(z),$$

and

$$\Omega_{0,z}^{1,p}f(z)=\frac{zf'(z)}{p}.$$

Motivated by the classes $S_{b,p}^{*}(\phi)$ and $C_{b,p}(\phi)$ which were studied by Ali et al. [1], we introduce a more general class of complex order $S_{b,p,\beta}^{\lambda}(\phi)$ which we define in the following.

Definition 1.2. Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the open unit disk $\mathcal{U}$ onto a region in the right half-plane and symmetric with respect to the real axis, $\phi(0)=1$ and $\phi'(0)>0$. A functions $f(z)\in A(p)$ is in the class $S_{b,p,\beta}^{\lambda}(\phi)$ if

$$1+\frac{1}{b}\left\{\frac{1}{p}\,\frac{z\left(\Omega_{0,z}^{\lambda,p}f(z)\right)'+\beta z^2\left(\Omega_{0,z}^{\lambda,p}f(z)\right)''}{(1-\beta)\,\Omega_{0,z}^{\lambda,p}f(z)+\beta\left(\Omega_{0,z}^{\lambda,p}f(z)\right)'}-1\right\}\prec\phi(z), \tag{1.8}$$

where $b\in\mathbb{C}\backslash\{0\}$, $0\le\beta\le1$, $0\le\lambda<1$, $p\in\mathbb{N}$ and $z\in\mathcal{U}$. Also, we let $S_{1,p,\beta}^{\lambda}(\phi)=S_{p,\beta}^{\lambda}(\phi)$.

The above class $S_{b,p,\beta}^{\lambda}(\phi)$ is of special interest and it contains many well-known classes of analytic functions. In particular; for $\lambda=0$ and $\beta=0$, we have

$$S_{b,p,0}^{0}(\phi)=S_{b,p}^{*}(\phi)$$

where $S_{b,p}^{*}(\phi)$ is precisely the class which was studied by Ali et al. [1], while for $\lambda=0$ and $\beta=1$, we have

$$S_{b,p,1}^{0}(\phi)=C_{b,p}(\phi)$$

where $C_{b,p}(\phi)$ is precisely the class which was introduced by Ali et al. [1].

Furthermore, by specializing the parameters λ,b,p and β we obtain the following subclasses which were studied by various others:

1- For $\lambda=0,\ b=1\ ,p=1$ and $\beta=0$, we get the class $S_{1,1,0}^{0}(\phi)=S^{*}(\phi)$ which was studied by Ma and Minda [2].

2- For $\lambda = 0,\ b = 1, p = 1$ and $\beta = 1$, we get the class $S^0_{1,1,1}(\phi) = C(\phi)$ which was studied by Ma and Minda [2].

3- For $\lambda = 0, p = 1$ and $\beta = 0$, we have the class $S^0_{b,1,0}(\phi) = S^*_b(\phi)$ which was studied by Ravichandran et al. [5].

4- For $\lambda = 0, p = 1$ and $\beta = 1$, we have the class $S^0_{b,1,0}(\phi) = C_b(\phi)$ which was studied by Ravichandran et al. [5].

5- For $\lambda = 0,\ b = 1$ and $\beta = 0$, we get the class $S^0_{1,p,0}(\phi) = S^*_p(\phi)$ which was studied by Ali et al. [1].

Very recently, Ali et al. [1] obtained the sharp coefficient inequalities for functions in the class $S^*_{b,p}(\phi)$ and many other subclasses of $A(p)$.

In the present paper, we obtain Fekete-Szegö inequalities of the functions belonging to the classes $S^\lambda_{1,p,\beta}(\phi)$ and $S^\lambda_{b,p,\beta}(\phi)$. These results are extended to the other classes that were defined earlier. See [1], [2] and [5] for Fekete-Szegö problem for certain related classes of functions.

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \cdots$$

in the open unit disk $\mathcal{U}$ satisfying the condition $|w(z)| < 1$. In order to prove our main results, we need the following lemmas which shall be used in the sequel.

Lemma 1.3 [1]. If $w \in \Omega$, then

$$|w_2 - t w_1^2| \le \begin{cases} -t & if \quad t \le -1, \\ 1 & if \quad -1 \le t \le 1, \\ t & if \quad t \ge 1. \end{cases}$$

when $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if

$$w(z) = z\frac{\lambda + z}{1 + \lambda z}, \qquad (0 \le \lambda \le 1)$$

or one of its rotations, while for $t = 1$, the equality holds if and only if

$$w(z) = -z\frac{\lambda + z}{1 + \lambda z}, \qquad (0 \le \lambda \le 1)$$

or one of its rotations .

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Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:

$$|w_2 - tw_1^2| + (t+1)|w_1|^2 \leq 1\,, \qquad (-1 < t \leq 0)$$

and

$$|w_2 - tw_1^2| + (1-t)|w_1|^2 \leq 1\,, \qquad (0 < t < 1).$$

Lemma 1.4 [3, inequality 7, p.10]**.** If $w \in \Omega$, then for any complex number t,

$$|w_2 - tw_1^2| \leq \max(1, |t|).$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

1- Coefficient bounds

By making use of Lemmas 1.4-1.5, we prove the following:

Theorem 2.1. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_n 's are real with $B_1 > 0, B_2 \geq 0$, and θ is a real number and

$$\sigma_1 = \frac{\varphi_1^2(1+\beta p)^2[(B_2 - B_1) + pB_1^2]}{2\varphi_2 pB_1^2[(1+\beta p)^2 - \beta^2]}\,, \tag{2.1}$$

$$\sigma_2 = \frac{\varphi_1^2(1+\beta p)^2[(B_2 + B_1) + pB_1^2]}{2\varphi_2 pB_1^2[(1+\beta p)^2 - \beta^2]}\,, \tag{2.2}$$

$$\sigma_3 = \frac{\varphi_1^2(1+\beta p)^2[B_2 + pB_1^2]}{2\varphi_2 pB_1^2[(1+\beta p)^2 - \beta^2]}\,. \tag{2.3}$$

If $f(z)$ given by (1.1) belongs to the class $S_{p,\beta}^{\lambda}(\phi)$ and φ_1, φ_2 given by (1.7), then

$$|a_{p+2} - \theta a_{p+1}^2| \leq \begin{cases} \dfrac{p}{2\varphi_2}\left(\dfrac{1+\beta(p-1)}{1+\beta(p+1)}\right)\left\{B_2 + pB_1^2\left[1 - \dfrac{2\theta\varphi_2}{\varphi_1^2}\left(1 - \dfrac{\beta^2}{(1+\beta p)^2}\right)\right]\right\}, & \theta \leq \sigma_1, \\ \dfrac{pB_1}{2\varphi_2}\left(\dfrac{1+\beta(p-1)}{1+\beta(p+1)}\right) & ,\sigma_1 \leq \theta \leq \sigma_2, \\ \dfrac{p}{2\varphi_2}\left(\dfrac{1+\beta(p-1)}{1+\beta(p+1)}\right)\left\{-B_2 + pB_1^2\left[\dfrac{2\theta\varphi_2}{\varphi_1^2}\left(1 - \dfrac{\beta^2}{(1+\beta p)^2}\right) - 1\right]\right\}, & \theta \geq \sigma_2. \end{cases} \tag{2.4}$$

Further, if $\sigma_1 \leq \theta \leq \sigma_3$, then

$$|a_{p+2} - \theta a_{p+1}^2| + \frac{\varphi_1^2(1+\beta p)^2}{2\varphi_2 pB_1[(1+\beta p)^2 - \beta^2]}\left\{1 - \frac{B_2}{B_1} + \left[\frac{2\theta\varphi_2}{\varphi_1^2}\left(1 - \frac{\beta^2}{(1+\beta p)^2}\right) - 1\right]pB_1\right\}|a_{p+1}|^2 \leq \frac{pB_1}{2\varphi_2}\left(\frac{1+\beta(p-1)}{1+\beta(p+1)}\right) \tag{2.5}$$

If $\sigma_3 \leq \theta \leq \sigma_2$, then

$$\left|a_{p+2}-\theta a_{p+1}^2\right|+\frac{\varphi_1^2(1+\beta p)^2}{2\varphi_2 pB_1[(1+\beta p)^2-\beta^2]}\left\{1+\frac{B_2}{B_1}-\left[\frac{2\theta\varphi_2}{\varphi_1^2}\left(1-\frac{\beta^2}{(1+\beta p)^2}\right)-1\right]pB_1\right\}\left|a_{p+1}\right|^2$$
$$\leq \frac{pB_1}{2\varphi_2}\left(\frac{1+\beta(p-1)}{1+\beta(p+1)}\right) \tag{2.6}$$

For any complex number,

$$\left|a_{p+2}-\theta a_{p+1}^2\right| \leq \frac{pB_1}{2\varphi_2}\left(\frac{1+\beta(p-1)}{1+\beta(p+1)}\right)\max\left\{1,\left|\frac{2\theta\varphi_2}{\varphi_1^2}\left(1-\frac{\beta^2}{(1+\beta p)^2}\right)pB_1-\frac{B_2}{B_1}-pB_1\right|\right\} \tag{2.7}$$

The results are sharp.

Proof. If $f(z)\in S_{p,\beta}^{\lambda}(\phi)$, then there is a Schwarz function

$$w(z)=w_1z+w_2z^2+\cdots \ \in\Omega$$

such that

$$\frac{1}{p}\,\frac{z\left(\Omega_{0,z}^{\lambda,p}f(z)\right)'+\beta z^2\left(\Omega_{0,z}^{\lambda,p}f(z)\right)''}{(1-\beta)\,\Omega_{0,z}^{\lambda,p}f(z)+\beta\left(\Omega_{0,z}^{\lambda,p}f(z)\right)'}=\phi\big(w(z)\big) \tag{2.8}$$

since

$$\frac{1}{p}\,\frac{z\left(\Omega_{0,z}^{\lambda,p}f(z)\right)'+\beta z^2\left(\Omega_{0,z}^{\lambda,p}f(z)\right)''}{(1-\beta)\,\Omega_{0,z}^{\lambda,p}f(z)+\beta\left(\Omega_{0,z}^{\lambda,p}f(z)\right)'}=1+\frac{(1+\beta p)}{p[1+\beta(p-1)]}\varphi_1 a_{p+1}z+$$
$$+\left[\frac{2}{p}\left(\frac{1+\beta(p-1)}{1+\beta(p+1)}\right)\varphi_2 a_{p+2}-\frac{(1+\beta p)^2}{p[1+\beta(p-1)]^2}\varphi_1^2a_{p+1}^2\right]z^2+\cdots \tag{2.9}$$

We have from (2.8),

$$a_{p+1}=\frac{p[1+\beta(p-1)]B_1w_1}{\varphi_1(1+\beta p)}, \tag{2.10}$$

and

$$a_{p+2}=\frac{p}{2\varphi_2}\left(\frac{1+\beta(p-1)}{1+\beta(p+1)}\right)\{B_1w_2+(B_2+pB_1^2)w_1^2\} \tag{2.11}$$

Therefore, we have

$$a_{p+2}-\theta a_{p+1}^2=\frac{pB_1}{2\varphi_2}\left(\frac{1+\beta(p-1)}{1+\beta(p+1)}\right)\{w_2-vw_1^2\} \tag{2.12}$$

where

$$v:=\left[\frac{2\theta\varphi_2}{\varphi_1^2}\left(1-\frac{\beta^2}{(1+\beta p)^2}\right)-1\right]pB_1-\frac{B_2}{B_1} \tag{2.13}$$

The results (2.4)-(2.7) are established by an application of Lemma 1.3 and inequality (2.7) by Lemma 1.4.

To show that the bounds in (2.4)-(2.7) are sharp, we define the functions $K_{\phi n}$ $(n=2,3,\dots)$ by

$$\frac{1}{p}\frac{z\left(\Omega_{0,z}^{\lambda,p}K_{\phi n}(z)\right)' + \beta z^2\left(\Omega_{0,z}^{\lambda,p}K_{\phi n}(z)\right)''}{(1-\beta)\,\Omega_{0,z}^{\lambda,p}K_{\phi n}(z) + \beta\left(\Omega_{0,z}^{\lambda,p}K_{\phi n}(z)\right)'} = \phi(z^{n-1}), \qquad K_{\phi n}(0) = \left(K_{\phi n}\right)'(0) - 1 = 0$$

and the functions F_r , G_r $(0 \le r \le 1)$ defined by

$$\frac{1}{p}\frac{z\left(\Omega_{0,z}^{\lambda,p}F_r(z)\right)' + \beta z^2\left(\Omega_{0,z}^{\lambda,p}F_r(z)\right)''}{(1-\beta)\,\Omega_{0,z}^{\lambda,p}F_r(z) + \beta\left(\Omega_{0,z}^{\lambda,p}F_r(z)\right)'} = \phi\left(\frac{z(z+r)}{1+rz}\right), \qquad F_r(0) = F_r{}'(0) - 1 = 0$$

and

$$\frac{1}{p}\frac{z\left(\Omega_{0,z}^{\lambda,p}G_r(z)\right)' + \beta z^2\left(\Omega_{0,z}^{\lambda,p}G_r(z)\right)''}{(1-\beta)\,\Omega_{0,z}^{\lambda,p}G_r(z) + \beta\left(\Omega_{0,z}^{\lambda,p}G_r(z)\right)'} = \phi\left(-\frac{z(z+r)}{1+rz}\right), \qquad G_r(0) = G_r{}'(0) - 1 = 0$$

respectively, it is clear that the functions $K_{\phi n}$, F_r and G_r belong to the class $S_{p,\beta}^{\lambda}(\phi)$. If $\theta < \sigma_1$ or $\theta > \sigma_2$, then the equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. If $\sigma_1 < \theta < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\theta = \sigma_1$, then the equality holds if and only if f is F_r or one of its rotations. If $\theta = \sigma_2$, then the equality holds if and only if f is G_r or one of its rotations.

Theorem 2.2. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, where B_n 's are real with $B_1 > 0$ and $B_2 \ge 0$.

If $f(z)$ given by (1.1) belongs to the class $S_{b,p,\beta}^{\lambda}(\phi)$ and φ_1, φ_2 given by (1.7), then for any complex number θ, we have

$$\left|a_{p+2} - \theta a_{p+1}^2\right| \le \frac{p|b|B_1}{2\varphi_2}\left(\frac{1+\beta(p-1)}{1+\beta(p+1)}\right)\max\left\{1, \left|\left[\frac{2\theta\varphi_2}{\varphi_1^2}\left(1 - \frac{\beta^2}{(1+\beta p)^2}\right) - 1\right]pB_1 - \frac{B_2}{B_1}\right|\right\} \tag{2.14}$$

The result is sharp.

Proof. The proof is similar to the proof of Theorem 2.1.

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