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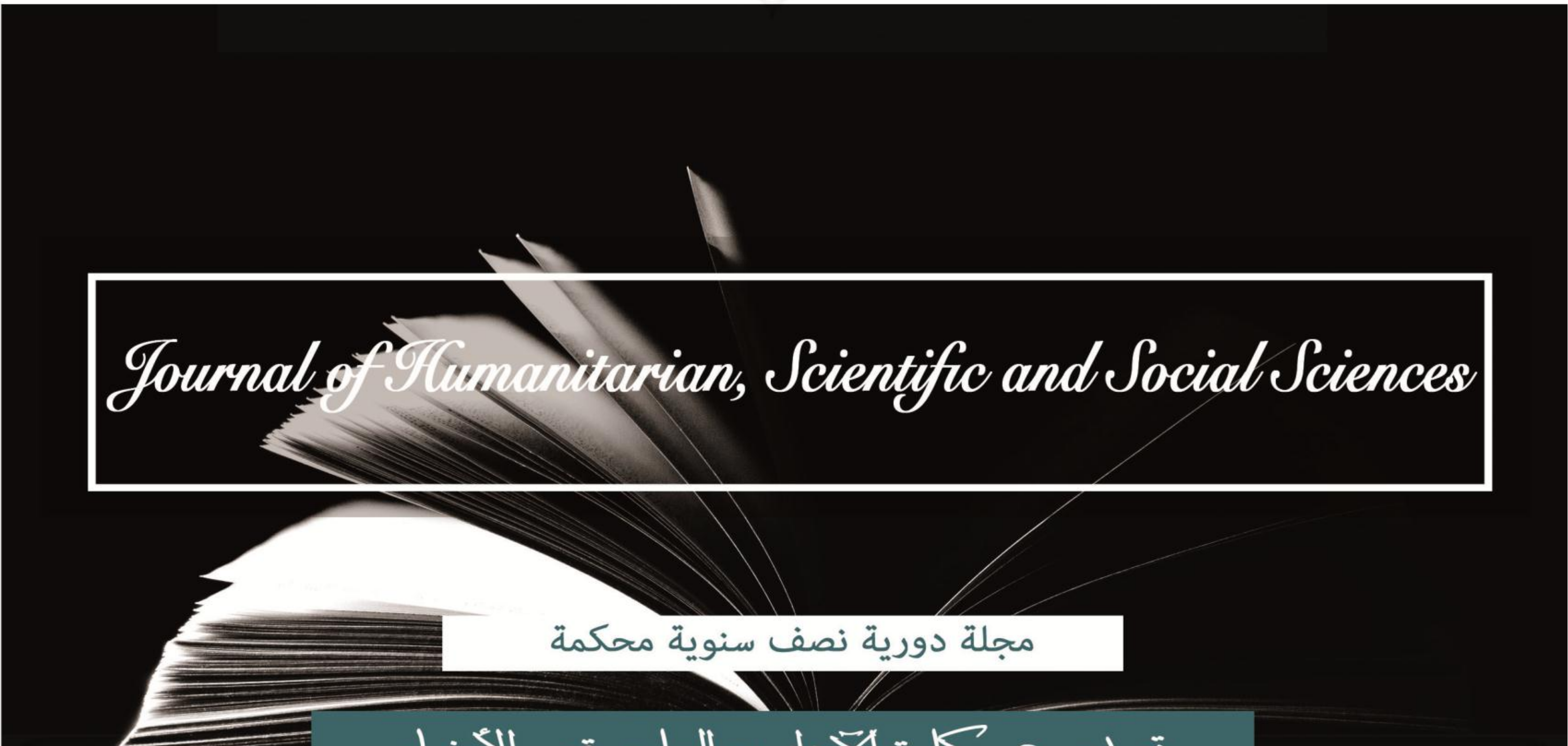

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ISSUE INDEX

English for Academic Writing Purposes

An Empirical Analysis of Needs and Wants

That Face Libyan EFL Learners atTertiary Levels

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Abstract

Academic writing remains an essential skill for English learners in Libya in EFL context. English courses offered in Libya at tertiary levels mostly focus on teaching for general purposes, few concerns writing for academic purposes, even for English majors. Therefore, This paper examines the needs and lack of EFL Libyan learners who study English for academic writing purpose . Analysis of the questionnaire and learners' texts that have been used to collect data were presented. Results indicate that most of the participants have never taken an academic English course before and that learners found a huge difficulty in academic writing skills. Therefore, an EAWP (English for Academic Writing purpose) course is needed for learners who study at tertiary levels in Libya in order to write academic papers appropriately.

Introduction*:*

In recent years, academic writing has become an important tool for people in today's international community. Therefore, it is used in reporting analyses of current events for newspaper, composing academic essays, letters and e-mail messages. The ability to write effectively allows individuals from different cultures and backgrounds to communicate. In addition, it is widely recognized that writing plays a vital role not only in conveying information, but also in transforming knowledge to create new knowledge. Many believe that teaching academic writing for specific purposes will benefit learners more than teaching it for general purposes, because that enhance students' needs and necessities. In this context, program can be designed by analysing learners' needs and identifying some certain points which learners struggle with when they write academically. Needs analysis can be defined as "lacks rather than needs that come to determine curriculum since what we are really interested in is the gap between the target proficiency and the present proficiency of the learners" (Basturkmen 1998 p.1). This research paper will focus on analysing the needs and wants of group of learners who study English for academic writing purpose. This will be discussed by presenting the description of the teaching context, discussion of needs analysis tools and information about learners' present difficulties whichwere collected through questionnaire and writing texts' analysis. Following that, findings of learners' needs and implications of this project will be presented.

Teaching context:

These are EFL Libyan learners who were involved in an intensive teaching program for English Academic Writing Purpose (EAWP) which will be taken for fifteen weeks and three days per week and two hours per day.The aim of this program is to assist learners who have been asked to write an academic essay in order to complete their tertiary studies in overseas countries. These learners study at a private institution in Libya and they are a mixed group (males &females) who are aged between 18-21 years. Their first language is Arabic and they have Arabic cultural backgrounds as well as they have already learnt English language in both high school and university for three years through using the grammar translation method. This method basically depends on translating the language from English language to Arabic language and teaching how to write some simple and compound sentences with certain grammatical features. Therefore, the learners have not benefited from such method, because they have not developed their academic writing abilities.

Moreover, these learners have various levels of English proficiency, but the focus in this context will be on an intermediate level. This is because of the fact that, although they have good levels of general English writing proficiency, they still in need to improve their academic writing abilities,since they have not been introduced to how use variety types of sentences structures. Also, they in need to use technical and formal vocabulary in their writing. This program will assist these learners by identifying the wants, necessities and the gap between their present learning situations and their target situations. Therefore, the aim of this research paper is to help learners with their academic essays, so they could write in more a proficient manner and complete their tertiary studies in overseas countries.

Discussion of previous studies and needs analysis :

As it has been stated in the previous studies, it is noted that needs analysis can be divided into three categories that necessities, lacks and wants (Hutchinson and Waters cited in Benesch, 2001). Another study includes that needs analysis “emphasize the important of investigating the competencies, strengths and weaknesses that students have prior to the beginning of a course of study, and provide arrange of devices that could be used for this purpose” (Richterich and Chancerel cited in Tajino, James & Kijima 2004 p. 2). Also, it is confirmed that giving students examples of strategies for improving, planning, organizing, drafting and editing would develop the necessities of student's academic witting. Furthermore, teachers should evaluate student' writing to know their common grammatical structure and syntactic errors (Giridharan 2012).

Moreover, a study by Sanders (2006) includes that a questionnaire was developed in EAP course to identify certain elements which are important for students' learning needs such as paraphrasing and paragraph structures. Also, a study by Vardi (2002) demonstrates that learners' needs appear when those learners are required to write long complex texts objectively and explicitly which they may not have experienced before. Also, a study by Barkaoui (2007) confirms that raising learners' awareness about successful writing processes is important by providing them with models and clear specific learning goal. For example, it is concluded that vocabulary is more important to maintain a consistent academic pattern (Swales &Feak 2004). These studies seems to suggest that needs analysis means to identify learners' problems which they may have at the beginning of the course and focus on these problems with the aim that provides a good support to assist students' learning. Thus, this context has focused on identifying the problems which learners have and providing them with good writing strategies to overcome their problems.

Furthermore, information about need analysis can be gathered by different ways such as questionnaire and analysis of authentic written texts (Evans & St John 1998). In this context, there are two methods which were used to gather information about the learners' needs; primarily a questionnaire was given to 15 learners to gather information about their needs or necessities. According toBoshr and Smalkoski (2002), a questionnaire has been used to collect information about learners' needs regarding course design for EAP. Secondly, analysis of learner's writing texts were used to identify the microstructure level of the texts and identify the gap between the current levels of learners and their target situations. For example, Lewin and Fine &Young (2001)include that analysis of genre texts can identify a gap between the current level of learners and the target situation. Based on the above studies, it can be confirmed that the methods of texts' analysisand the questionnaire which have been used in this program would be very useful ways to identify the academic writing' gap between what do learners currently can do and what do learners can do after completing their academic stages.

- Research Data:

1. Analysis of the questionnaire:

A questionnaire was one of the methods which were used to gather information about the learners' needs and necessities in this context. Learners were asked to answer four questions that were written by English language regarding the difficulties that face learners while they are writing an academic essay. The questionnaire was distributed equally to all 15 learners who study at an English collage to ensure fairness and valid data for better learning program.

First Question:[see Appendix]

Decide which of these sentences are the topic sentence and decide the order of the supporting sentences and number them .

Percentage%		Incorrect answer	Correct answer	Total
Incorrect	correct			
80	20	12	3	15

This table shows that learners have faced a lot of problems with writing and organizing the correct form of the paragraph. That is attributed to the students' disability to distinguish between the topic , supporting and concluding sentences. For the first question, the majority of the learners' responses about 80% are incorrect . On the other hand, about 20% of the learners' responses are correct. As a result, It is noted that learners in this context encountered difficulties with organizing the different parts of the paragraph.

Second Question: **[see Appendix]**

Choose the transition signal that best shows the relationship between the following sentences.

Percentage%		Incorrect answers	Correct answers	Total
incorrect	correct	10	5	15
66.6	33.3			

From this table, it is resulted that about 66.6% of the learners faced difficulties in determining the best transition signals but about 33.3% of them have not faced these problems. The reason for these errors is that most learners did not know how to link between different parts of the paragraph by using the suitable transition signals.

Third question:[see Appendix]

Do you have difficultiesin constructing compound and complex sentences when you write an academic paragraph ?

Yes	80%
No	20%

Based on the table above, it shows that most of the learners about 80% have confirmed that they faced problems in constructing compound and complex sentences while they are writing their paragraphs whereas about 30% of the learners found it easy to use these types of the sentences in their writing. learners need to know how to use complex sentences in their writing rather only using simple or compound sentences, so later on they can perform well in the complex context of academic writing in their tertiary studies.

Forth Question: [see Appendix]

Do you think, you can produce new and technical words quickly when writing an academic essay ?

Yes	80%
No	20%

In the final question, a wide range of academic vocabulary is another problem that has faced the learners. Difficulty in choosing the correct vocabulary even with theuse of a dictionary was another aspect that faced the learners in this context. About 80% of the learners faced difficulties in producing new words while they are writing an academic paragraph whereas about 20% of the learners have not faced those problems.

2. Analysis of written texts*:*

Secondly, analysis of learners' writing was used to gather information about the microstructure level of the learners' texts and identify the problems that the learners have with organizing and writing each part of their essays' paragraphs. For instance, a study by Burstein and Marcu (2003) states that essay topics have been evaluated to collect data about learners' weaknesses in order to seek students' needs. Therefore, in this context, learners' essays have been evaluated to identify their weaknesses and the gap between their current levels and their target situations. The learners attempt to write essays about (*Rich countries should give aid to poor countries*). The analysis of the learners' essays resulted in to the following:

Firstly, the learners have difficulties in using technical and academic vocabulary and this can be seen through skimming learners' essays. As a result, it is suggested that English learners need to be taught how to gain academic vocabulary pattern, so that they will be able to practice writing which is one of the language skills. Therefore, this means that learners in this program should gain rich and technical vocabulary. Therefore, they can use more than one way to express their ideas in academic writing skill.

Second difficulty which learners need to improve in order to write a good academic essay is paragraph structures ((Lewin , Fine &Young 2001). In this context, most of the learners have a huge difficulty in constructing the topic sentences which is the important part of the paragraph. Also, learners have not stated the main idea, concluding sentences and recommendation of the essay very clearly. For example,in the first paragraph of one of thelearner's texts (*aid helps developing countries to improve medi*cal *treatment).* It is noticed that, there is a lack in this sentence to convey meaning to the reader and this writer should include more details in the sentence, so the main idea of the paragraph can be understood.

Thirdly, most learners' texts in this study have grammaticalerrors especially, in using of variety of sentences structures. For example, in the first paragraph regard one of the learner's text, it is stated that (*As result, life expectancy was increased*). Thus, it is noticed that this learner uses simple sentences in writing an essay instead of complex sentences. Based on the above results which have been taken from the analysis of the learners' texts, it can be seen that there are some certain grammatical features which learners had not paid attention to learn in the past. As result, there is a gap which can be defined as certain hidden skills which learnerscan not master in the current situation, so if these skills are identified, the learners will improve their academic writing abilities.

In addition, regarding the learners' texts, there is not linkage between different parts in each paragraph which make the paragraphs incoherent. learners have not used transition signals to link sentences within the paragraphs. For example, a study by Swales &Feak (2012) includes that to write academically, you should make many considerations which are audience, purpose, organization and flow of the writing. Thus, it can be inferred that, there is not linkage in the learners'texts and the learners should organise their writing with many considerations in order to write an academic essay.Consequently, learners' needs and wants should be taken into account for designing an effective EAWP program.

Finding:

In sum, discussion of needs analysis tools (questionnaire & learners' writing texts analysis) and relevant literature have revealed that EFL Libyan students who were involved in the EAWP course, had the most difficulties in the following:

- Using a wide range of vocabulary that assists learners to express their ideas in more than one way.
- Being able to write a good academic structured paragraph with well organised ideas using topic, supporting and concluding sentences.
- Being able to use and identify grammatical features as well as using varieties of sentences in writing such as compound and complex sentences.
- Being able to write academically and coherently with and between paragraphs by usingtransition signals to link different parts .

Implication:

Analysis of learners' needs should be starting point for designing a better program that satisfies learners' needs and necessities. According to Hamp-lyons (cited in Tajino, James & Kijima 2004), an EAWP program is starting point because it begins with a learner and a situation. This study confirms thatan EAWP course starts with analysing learners' needs that are regarding learning situation. Therefore, in this context, the EAWP course should start with learners' needs as a first step that makes this program more effective. For example, a study by West (cited in Bosher&Smallkoski 2004) indicates that learners' needs analysis must be translated into appropriate course objectives. This study seems to suggest that the EAWP course must focus on the needs and necessities which learners have difficulties with by translating these needs and necessities as objectives of this course. Thus, according to the above needs analysis and finding of this project, it can be suggested that needs analysis is considered as an essential step in the EAWP course design.

Moreover, materials, books and methods of teaching in this program should be chosen regarding learners' needs (objectives of the course). For example, Bosherand Smallkoski (2004) include that materials and methods of teaching when designing an EAP course must be selected according to the results of the finding. This study demonstrates that when designing a course, teachers should select the relevant materials and approaches which will focus and assist learners' needs to improve their language learning. Consequently, in this context, teachers should take the above point into account, so learners will reach the aimed outcome and teachers would have a successful teaching program.

Next, teachers should provide consultation time for learners, so each learner obtains an opportunity to mention their needs and necessities. For example, conferences which can be defined as an appointment which can be organized between a teacher and a learner to discuss a particular piece of work or the process of learning or both of them (Brown & Hudson 1998).

This study defines the conferences as a plan for meeting that can be organized between a teacher and a student to discuss the issues regarding learning process. This approach will be very helpful for learners in this context, because by such way those learners will have the opportunities to meet the teacher face to face and discuss the certain points which they struggle with when constructing an academic essay writing. Therefore, such a useful strategy should be included in EAWP course.

In addition, there are a number of contextual factors that should be taken into account when designing the EWAP course. Firstly, learners in this context come from an Arabic language environment and this should be considered, because learners will be using English only at the institution while using the Arabic language in their daily life. Therefore, in such that case, teachers might suggest English writing program outside the classrooms, so learners can use more English every day out and inside the classrooms. Secondly, class size also should be considered when designing the EWAP course, since it is sometimes impossible to teach writing skills to a class of 30 learners. Also, this may have a great impact on the conferences strategy which has been mentioned above. This is because of that, if the class have 30 learners, a teacher can not provide enough time for each student. Thus, enough time should be considered for each lesson, so learners can ask some questions about their difficulties regarding certain points and teachers can give them the feedbacks, because it is critical for developing better content structure and overall language proficiency (Giridharan 2012) .

Finally, learners' motivation should be considered too. For example, it is claimed that students' motivation should be considered when designing an EAP course because this helps teachers to work out what is needed to enable learners to reach the target aim (Evans & St John 1998). This study means that learners' motivation decide whether learners' needs become explicit or implicit which helps teacher to identify the learners' needs and assist them. Thus, teachers should motivate students to elicit their difficulties, so teachers help them to overcome their problems. Based on the above studies, all stakeholders should consider the above factors when designing an English writingcourse, so they can reach their aims and design a successful teaching program.

Conclusion:

To sum up, it has been indicated that descriptions of target context, discussion of needs analysis tools, analysis of the questionnaire and learners' texts that have been used to collect data which was presented. Then, finding of Libyan learners' needs and some suggestion for the EAWP course have been presented. Results of this study have focused on needs and necessities which learners who study in the EAWP course find difficulties with. learners in this context should be though academic writing skill with considerations of teaching specific aspects that are main structures of the paragraph, technical and new vocabulary, using transition signals and using different types of the sentences. Also, It can be recommended that teachers and learners should participate in designing an effective academic course by both ways, firstly, student should ask explicit questions about their weaknesses to be understood. Secondly, teachers should provide a good explicit answers or feedbacks for students. Thus, it can be ensured that learners' necessities have been identified and resulted into an effective teaching and learning program.

Reference list:

Basturkmen, H . (1998). Refining procedures: a needs analysis project at Kuwait University.

Forum[Online version],36(4). 1-18. Avalable on
http://exchanges.state.gov/forum/vols/vol36/no4/p2.htm
(accessed 5 Agust 2007)
Baynham, M (2000) 'Academic writing in new and emergent discipline areas', in lea, M R&

Stierer,B (eds.), Student writing in higher education: new contexts, Open University

press, Buckingham,pp. 17-31.

Bosher, S &Smalkoski, K. (2002) From needs analysis to curriculum development: designing a course in health-care communication for immigrant students in the USA. *English for Academic Purposes,* 21(1). 59-79. Pergamion database.

Brown, J & Hudson, T. (1998) 'The alternatives in language assessment', *TESOL Quarterly*, 32(4): 653-675.

Burstein, J &Marcu, D. (2003). A machine learning approach for identification of thesis and conclusion statements in student essays. *Computers and the Humanities,* 37(4), 455-467. ProQuest 5000 Database.

Dudley-Evans, T & St John, M (1998). *Developments in English for Specific Purposes*. Cambridge: CambridgeUniversity Press.

Lewin, B, Fine, J & Young, L (2001) 'A critical review of genre analysis', Expository Discourse: A Genre-based Approach to Social Science Research Texts, Continuum, London: 1-22.

Sanders, F (2006).The relevance of EAP for success at university: a study of students

perspectives. *TESOL department*. Melbourne, University of Melbourne.

Master degree: 112.

Swales, J &Feak, C. (2004). *Academic writing for graduate students: essential tasks and skills***.** Ann Arbor: University of Michigan Press.

Tajino, A, James, R & Kijima, K. (2005). Beyond needs analysis: soft systems methodology for meaningful collaboration in EAP course design. *Journal of English for Academic purposes,*4 (1), 27-42, ScienceDirect Databases.

Vardi, I (2002). *How do first year lectures help students develop writing skills?* HERDSA 2002 conference, CurtinUniversity.

Beena,G (2012).*China Education Review: Identifying Gaps in Academic writing of ESL students* .Curtin University, Miri , Malaysia.

Barkaoui. K (2007) . *TESOL Reporter* .Teaching Writing to Second Language Learners: Insights from Theory and Research. University of Toronto, Canada.

Swales.J&Feak.CH(2012). Academic writing for Graduate students: Essential skills and tasks. *An Approach to Academic writing*. Http://www.press.umich.edu/titleDetailDesc.do.

Appendix

The following questionnaire was distributed equally to all 15 Libyan learners who study at an English collage:

1- **Decide which of the following sentences is the topic sentences of the paragraph and decide the order of the supporting sentences and number them[ss1,ss2,ss3 and so on….]**

a- It enables customers to do several blanking transactions twenty-hours a day…..
b- In addition, a customer can transfer funds between accounts or get a cash advance on a credit card…….
c- An automated teller machine [ATM] is a convenient miniature bank……..
d- For example, a customer can use an ATM to deposit money and with draw a limited of cash…….

2- Choose the transition signal that best shows the relationship between these sentences in each group from the choices given in the parentheses, and add punctuation and change capital to small letters if necessary.

1- The same article said that the causes of inflation……..the cure for inflation was not so easy to prescribe. [however, for example, therefore]
2- Era also suggested that rising wages were one of the primary causes of inflation………the government should take action to control wages. [however, for example].
3- In physics, the weighs of an object is the gravitational force with which the earth attracts it ………if a man weighs 150 pounds, this means that the earth pulls him down with a force of 150 pounds.

3- Do you have difficulties in constructing compound and complex sentences when you write an academic paragraph ?

-Yes

-No

4- Do you think, you can produce new and technical words quickly when writing an academic essay ?

-Yes

-No

FRACTIONAL DERIVATIVE OPERATORS ASSOCIATED WITH SOME SUBCLASSES OF STARLIKE FUNCTIONS

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Abstract

In this paper we introduce new subclasses of starlike functions in the open unit disk by using fractional derivative operator. We obtain coefficient inequalities and distortion theorems for functions belonging to these subclasses. Further results include distortion theorems (involving the generalized fractional derivative operator). The radii of convexity for functions belonging to these subclasses are also studied.

Keywords: univalent functions; starlike functions; convex functions; fractional derivative operators.

Mathematics subject classification: 30C45, 26A33

1- Introduction

Two interesting subclasses $T^*(\alpha,\beta,\gamma)$ and $C(\alpha,\beta,\gamma)$ of univalent starlike functions with negative coefficients in the open unit disk $\mathcal{U}$ were introduced by Srivastava and Owa[10]. These classes become the subclasses of the class $K(\alpha,\beta)$ which was introduced by Gupta [1] when the function $f(z)$ is univalent with negative coefficients. Using the results of Srivastava and Owa[10], Srivastava and Owa[11] have obtained several distortion theorems involving fractional derivatives and fractional integrals of functions belonging to the classes $T^*(\alpha,\beta,\gamma)$ and $C(\alpha,\beta,\gamma)$.

Fractional calculus operators have recently found interesting applications in the theory of analytic functions (see e.g. [2, 4, 8, 9]). In the present paper, by making use of a certain fractional derivative operator, we introduce new subclasses $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ (defined below) of starlike functions with negative coefficients.

This paper is organized as follows: Section 2 gives preliminary details and definitions of starlike functions, convex functions and fractional derivative operators. In Section 3 we describe coefficient inequalities for the functions belonging to the subclasses $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. Section 4 considers the distortion properties. Its further distortion properties (involving the generalized fractional derivative operator) are discussed in section 5. Finally, in section 6 we determine the radii of convexity for functions belonging to these subclasses of starlike functions.

2- Preliminaries And Definitions

Let A denote the class of functions defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z: |z| < 1\}$. Further, let S denote the class of all functions in A which are univalent in $\mathcal{U}$. Then a function $f(z)$ belonging to the class S is said to be starlike of order α if and only if (see, e.g. , [2,5])

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ , \qquad z \in \mathcal{U} \tag{2.2}$$

for some $\alpha(0 \le \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α.A function $f(z)$ belonging to the class S is said to be convex of order α if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad z \in \mathcal{U} \tag{2.3}$$

for some $\alpha(0 \le \alpha < 1)$. We denote by $K(\alpha)$ the class of all convex functions of order α.Note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, and that,

$$S^*(\alpha) \subseteq S^*(0) = S^*,$$
$$K(\alpha) \subseteq K(0) = K,$$

and

$$K(\alpha) \subset S^*(\alpha) \subset S$$

for $0 \le \alpha < 1$.The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson[5].

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \ge 0) \tag{2.4}$$

We denote by $T^*(\alpha)$ and $C(\alpha)$, respectively, the classes obtained by taking the intersections of $S^*(\alpha)$ and $K(\alpha)$ with T, that is

$$T^*(\alpha) = S^*(\alpha) \cap T$$

and

$$C(\alpha) = K(\alpha) \cap T$$

The classes $T^*(\alpha)$ and $C(\alpha)$ were introduced by Silverman [6].

Let ${}_2F_1(a, b; c; z)$ be the Gauss hypergeometric function defined, for $z \in \mathcal{U}$ by; (see[7]).

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_{n} z^n}{(c)_n n!} \tag{2.5}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & , \quad n = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots.(\lambda+n-1) & , \quad n \in \mathbb{N} \end{cases}$$

for $\lambda \ne 0, -1, -2, \ldots$

Now we recall the following definitions of fractional derivative operators, adopted for working in classes of analytic functions in complex plane as follows see ([2,4]).

Definition 2.1. *The fractional derivative of order λ is defined by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \tag{2.6}$$

where $0 \leq \lambda < 1$, $f(z)$ *is analytic function in a simply- connected region of the z-plane containing the origin, and the multiplicity of* $(z-\xi)^{-\lambda}$*is removed by requiring log* $(z-\xi)$ *to be real when* $z-\xi > 0$.

Definition 2.2.*Let* $0 \leq \lambda < 1$, *and*$\mu, \eta \in \mathbb{R}$. *Then, in terms of the familiar Gauss's hypergeometric function* ${}_2F_1$ *, the generalized fractional derivative operator* $J_{0,Z}^{\lambda,\mu,\eta}$ *is*

$$J_{0,Z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\xi}{z}\right) d\xi\right) \tag{2.7}$$

where $f(z)$*is analytic function in a simply- connected region of the z-plane containing the origin with the order* $f(z) = O(|z|^{\varepsilon})$, $z \to 0$, *where*$\varepsilon > max\{0, \mu-\eta\} - 1$, *and the multiplicity of* $(z-\xi)^{-\lambda}$ *is removed by requiring log* $(z-\xi)$ *to be real when* $z-\xi > 0$.

Notice that

$$J_{0,Z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z), \qquad 0 \leq \lambda < 1 \tag{2.8}$$

Now we define the following classes of starlike functions based on fractional derivative operator.

Definition 2.3.*The function* $f(z) \in T$ *is said to be in the class* $T_{\lambda,\mu,\eta}^*(\alpha,\beta,\gamma)$ *if*

$$\left|\frac{\frac{z\left(P_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{g(z)} - 1}{\frac{z\left(P_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{g(z)} + 1 - 2\beta}\right| < \gamma \qquad (z \in \mathcal{U}) \tag{2.9}$$

$$(\lambda \geq 0, \mu < 2\,;\ \eta > \max(\lambda,\mu) - 2\,;\ 0 \leq \alpha < 1; 0 \leq \beta < 1;\ 0 < \gamma \leq 1)$$

for the function

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n (b_n \geq 0\,) \tag{2.10}$$

belonging to $T^*(\alpha)$. *Dented by* $P_{0,z}^{\lambda,\mu,\eta} f(z)$ *the modification of the fractional derivative operator which is defined in terms of* $J_{0,Z}^{\lambda,\mu,\eta}$ *as follows:*

$$P_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(2-\mu)\Gamma(2-\lambda+\eta)}{\Gamma(2-\mu+\eta)} z^{\mu} J_{0,Z}^{\lambda,\mu,\eta} f(z) \tag{2.11}$$

Further, if $f(z) \in T$ *satisfies the condition* (2.9) *for* $g(z) \in C(\alpha)$,*we say that*$f(z) \in C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$.

The above-defined classes $T_{\lambda,\mu,\eta}^*(\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ are of special interest and they contain well-known classes of analytic functions. In particular, in view of (2.11), we find that

$$P_{0,z}^{0,0,\eta} f(z) = f(z) \tag{2.12}$$

Thus, for $\lambda = \mu = 0$, we have

$$T^*_{0,0,\eta}(\alpha,\beta,\gamma) = T^*(\alpha,\beta,\gamma)$$

and

$$C_{0,0,\eta}(\alpha,\beta,\gamma) = C(\alpha,\beta,\gamma)$$

where $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ are the subclasses of starlike functions which were studied by Srivastava and Owa[10], andSrivastava and Owa[11].

In order to prove our results we shall need the following lemmas for the classes $T^*(\alpha)$and$C(\alpha)$due to Silverman [6]:

Lemma 2.4. *Let the function $g(z)$ defined by* (2.10). *Then $g(z)$ is in the class $T^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)b_n \le 1-\alpha \tag{2.13}$$

Lemma 2.5. *Let the function $g(z)$ defined by* (2.10). *Then $g(z)$ is in the class $C(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)b_n \le 1-\alpha \tag{2.14}$$

We mention to the following known result which shall be used in the sequel (see Raina and Srivastava[4]).

Lemma 2.6. *Let $\lambda,\mu,\eta \in \mathbb{R}$, such that $\lambda \ge 0$ and $k > max\{0,\mu-\eta\}-1$ then*

$$J_{0,Z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} z^{k-\mu} \tag{2.15}$$

3- Coefficient Inequalities

Theorem 3.1. *Let the function $f(z)$ be defined by* (2.4). *If $f(z)$belongs to the class $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$, then*

$$\sum_{n=2}^{\infty}\left[\psi(n)n(1+\gamma)a_n - \frac{(1-\alpha)(1-\gamma+2\gamma\beta)}{n-\alpha}\right] \le 2\gamma(1-\beta) \tag{3.1}$$

where

$$\psi(n) = \frac{(2)_{n-1}(2+\eta-\mu)_{n-1}}{(2-\mu)_{n-1}(2+\eta-\lambda)_{n-1}} \ , \qquad n \ge 2 \tag{3.2}$$

Proof. Applying Lemma 2.6, we have from (2.4) and (2.11) that

$$P_{0,z}^{\lambda,\mu,\eta} f(z) = z - \sum_{n=2}^{\infty} \frac{(2)_{n-1}(2+\eta-\mu)_{n-1}}{(2-\mu)_{n-1}(2+\eta-\lambda)_{n-1}} a_n z^n$$

Since$f(z) \in T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$, there exist a function $g(z)$belonging to the class $T^*(\alpha)$ such that

$$\left|\frac{z\left(P_{0,z}^{\lambda,\mu,\eta}f(z)\right)' - g(z)}{z\left(P_{0,z}^{\lambda,\mu,\eta}f(z)\right)' + (1-2\beta)g(z)}\right| < \gamma, \quad z \in \mathcal{U} \tag{3.3}$$

It follows from (3.3) that

$$\text{Re}\left\{\frac{\sum_{n=2}^{\infty}[\psi(n)na_n - b_n]\, z^{n-1}}{2(1-\beta) - \sum_{n=2}^{\infty}[\psi(n)na_n + (1-2\beta)b_n]\, z^{n-1}}\right\} < \gamma \tag{3.4}$$

Choosing values of z on the real axis so that $\frac{z\left(P_{0,z}^{\lambda,\mu,\eta}f(z)\right)'}{g(z)}$ is real, and letting $z \to 1^-$ through real axis, we have

$$\sum_{n=2}^{\infty}[\psi(n)na_n - b_n] \le \gamma\left\{2(1-\beta) - \sum_{n=2}^{\infty}[\psi(n)na_n + (1-2\beta)b_n]\right\}$$

or, equivalently,

$$\sum_{n=2}^{\infty}[\psi(n)n(1+\gamma)a_n - (1-\gamma+2\gamma\beta)b_n] \le 2\gamma(1-\beta) \tag{3.5}$$

Note that, by using Lemma 2.4, $g(z) \in T^*(\alpha)$ implies

$$b_n \le \frac{1-\alpha}{n-\alpha} \quad , \qquad n \ge 2 \tag{3.6}$$

Making use of (3.6) in (3.5), we complete the proof of Theorem 3.1. □

Corollary 3.2. *Let the function $f(z)$ be defined by* (2.4) *be in the class* $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. *Then*

$$a_n \le \frac{2\gamma(1-\beta)(n-\alpha) + (1-\alpha)(1-\gamma+2\gamma\beta)}{\psi(n)n(1+\gamma)(n-\alpha)}, \quad n \ge 2 \tag{3.7}$$

where $\psi(n)$ is given by (3.2). *The result* (3.7) *is sharp for a function of the form:*

$$f(z) = z - \frac{2\gamma(1-\beta)(n-\alpha) + (1-\alpha)(1-\gamma+2\gamma\beta)}{\psi(n)n(1+\gamma)(n-\alpha)} z^n, \qquad n \ge 2 \tag{3.8}$$

with respect to

$$g(z) = z - \frac{1-\alpha}{n-\alpha} z^n, \qquad n \ge 2 \tag{3.9}$$

Remark 1. Letting $\lambda = \mu = 0$, and $\alpha = 0$ in Corollary 3.2, we obtain a result was proved by [Gupta [1], Theorem 3].

In a similar manner, Lemma 2.5 can be used to prove the following theorem:

Theorem 3.3. *Let the function $f(z)$ be defined by* (2.4) *be in the class* $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. *Then*

$$\sum_{n=2}^{\infty}\left[\psi(n)n(1+\gamma)a_n - \frac{(1-\alpha)(1-\gamma+2\gamma\beta)}{n(n-\alpha)}\right] \le 2\gamma(1-\beta) \tag{3.10}$$

where $\psi(n)$ is given by (3.2).

Corollary 3.4. *Let the function $f(z)$ be defined by* (2.4) *be in the class* $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. *Then*

$$a_n \leq \frac{2\gamma(1-\beta)n(n-\alpha)+(1-\alpha)(1-\gamma+2\gamma\beta)}{\psi(n)n^2(1+\gamma)(n-\alpha)}, \quad n \geq 2 \tag{3.11}$$

where $\psi(n)$*is given by* (3.2). *The result* (3.11) *is sharp for a function of the form:*

$$f(z) = z - \frac{2\gamma(1-\beta)n(n-\alpha)+(1-\alpha)(1-\gamma+2\gamma\beta)}{\psi(n)n^2(1+\gamma)(n-\alpha)} z^n, \quad n \geq 2 \tag{3.12}$$

with respect to

$$g(z) = z - \frac{1-\alpha}{n(n-\alpha)} z^n, \quad n \geq 2 \tag{3.13}$$

4- Distortion Properties

Next, we state and prove results concerning distortion properties of $f(z)$ belonging to the classes $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$.

Theorem 4.1. *Let* $\lambda,\mu,\eta \in \mathbb{R}$ *such that*

$$\lambda \geq 0; \mu < 2; \eta \geq \lambda\left(\frac{\mu-3}{\mu}\right) \tag{4.1}$$

Also, let $f(z)$ *defined by* (2.4) *be in the class* $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. *Then*

$$|f(z)| \geq |z| - A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)|z|^2, \tag{4.2}$$

$$|f(z)| \leq |z| + A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)|z|^2, \tag{4.3}$$

$$|f'(z)| \geq 1 - 2A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)|z|, \tag{4.4}$$

and

$$|f'(z)| \leq 1 + 2A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)|z| \tag{4.5}$$

for $z \in \mathcal{U}$, *provided that* $0 \leq \alpha < 1$, $0 \leq \beta < 1$ *and* $0 < \gamma \leq 1$, *where*

$$A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma) = \frac{(2-\mu)(2+\eta-\lambda)\{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)\}}{4(2+\eta-\mu)(1+\gamma)(2-\alpha)} \tag{4.6}$$

The estimates for $|f(z)|$ *and* $|f'(z)|$ *are sharp.*

Proof. We observe that the function $\psi(n)$ defined by (3.2) satisfy the inequality $\psi(n) \leq \psi(n+1)$, $\forall n \geq 2$, provided that $\eta \geq \lambda\left(\frac{\mu-3}{\mu}\right)$, thereby, showing that $\psi(n)$ is non-decreasing. Thus under the conditions stated in (4.1), we have

$$0 < \frac{2(2+\eta-\mu)}{(2-\mu)(2+\eta-\lambda)} = \psi(2) \leq \psi(n), \quad \forall n \geq 2 \tag{4.7}$$

For $f(z) \in T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$, (3.5) implies

$$2\psi(2)(1+\gamma)\sum_{n=2}^{\infty} a_n - (1-\gamma+2\gamma\beta)\sum_{n=2}^{\infty} b_n \leq 2\gamma(1-\beta) \tag{4.8}$$

For $g(z) \in T^*(\alpha)$, Lemma 2.4 yields

$$\sum_{n=2}^{\infty} b_n \leq \frac{1-\alpha}{2-\alpha} \tag{4.9}$$

So that (4.8) reduces to

$$\sum_{n=2}^{\infty} a_n \leq \frac{(2-\mu)(2+\eta-\lambda)\{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)\}}{4(2+\eta-\mu)(1+\gamma)(2-\alpha)}$$

$$= A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma) \tag{4.10}$$

Consequently,

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \tag{4.11}$$

and

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n \tag{4.12}$$

On using (4.11), (4.12) and (4.10), we easily arrive at the desired results (4.2) and (4.3).

Furthermore, we note from (3.5) that

$$\psi(2)(1+\gamma)\sum_{n=2}^{\infty} na_n - (1-\gamma+2\gamma\beta)\sum_{n=2}^{\infty} b_n \leq 2\gamma(1-\beta) \tag{4.13}$$

which, in view of (4.9), becomes

$$\sum_{n=2}^{\infty} na_n \leq \frac{(2-\mu)(2+\eta-\lambda)\{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)\}}{2(2+\eta-\mu)(1+\gamma)(2-\alpha)}$$

$$= 2A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)(4.14)$$

Thus, we have

$$|f'(z)| \geq 1 - |z| \sum_{n=2}^{\infty} na_n \tag{4.15}$$

and

$$|f'(z)| \leq 1 + |z| \sum_{n=2}^{\infty} n\, a_n \tag{4.16}$$

On using (4.15), (4.16) and (4.14), we arrive at the desired results (4.4) and (4.5).

Finally, we can prove that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp by taking the function

$$f(z) = z - \frac{(2-\mu)(2+\eta-\lambda)\{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)\}}{4(2+\eta-\mu)(1+\gamma)(2-\alpha)} z^2 \tag{4.17}$$

with respect to

$$g(z) = z - \frac{1-\alpha}{2-\alpha} z^2 \tag{4.18}$$

This completes the proof of Theorem 4.1. □

Corollary 4.2. *Let the function* $f(z)$ *be defined by* (2.4) *be in the class* $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. *Then the unit disk* $\mathcal{U}$ *is mapped onto a domain that contains the disk* $|w| < r_1$, *where*

$$r_1 = 1 - \frac{(2-\mu)(2+\eta-\lambda)\{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)\}}{4(2+\eta-\mu)(1+\gamma)(2-\alpha)} \tag{4.19}$$

The result is sharp with the extremal function defined by (4.17).

Remark 2. Letting $\lambda = \mu = 0$, and $\alpha = 0$ in Theorem 4.1, we obtain a result was proved by [Gupta [1], Theorem 4].

Theorem 4.3. *Under the conditions stated in* (4.1), *let the function* $f(z)$ *defined by* (2.4)*be in the class* $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. *Then*

$$|f(z)| \geq |z| - B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)|z|^2, \tag{4.20}$$

$$|f(z)| \leq |z| + B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)|z|^2, \tag{4.21}$$

$$|f'(z)| \geq 1 - 2B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)|z|, \tag{4.22}$$

and

$$|f'(z)| \leq 1 + (p+1)B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)|z| \tag{4.23}$$

for $z \in \mathcal{U}$, *provided that* $0 \leq \alpha < 1$, $0 \leq \beta < 1$ *and* $0 < \gamma \leq 1$, *where*

$$B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma) = \frac{(2-\mu)(2+\eta-\lambda)\{4\gamma(1-\beta)(2-\alpha)+(1-\alpha)(1-\gamma+2\gamma\beta)\}}{8(2+\eta-\mu)(1+\gamma)(2-\alpha)} \tag{4.24}$$

The estimates for $|f(z)|$ *and* $|f'(z)|$ *are sharp.*

Proof. By using Lemma 2.5, we have

$$\sum_{n=2}^{\infty} b_n \leq \frac{1-\alpha}{2(2-\alpha)} \tag{4.25}$$

since $g(z) \in C(\alpha)$, the assertions (4.20), (4.21), (4.22) and (4.23) of Theorem 4.3 follow if we apply (4.25) to (3.5).The estimates for $|f(z)|$ and $|f'(z)|$ are attained by the function

$$f(z) = z - \frac{(2-\mu)(2+\eta-\lambda)\{4\gamma(1-\beta)(2-\alpha)+(1-\alpha)(1-\gamma+2\gamma\beta)\}}{8(2+\eta-\mu)(1+\gamma)(2-\alpha)} z^2 \tag{4.26}$$

with respect to

$$g(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2 \tag{4.27}$$

This completes the proof of Theorem 4.3. □

Corollary 4.4. *Let the function* $f(z)$ *be defined by* (2.4) *be in the class* $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. *Then the unit disk* $\mathcal{U}$ *is mapped onto a domain that contains the disk* $|w| < r_2$, *where*

$$r_2 = 1 - \frac{(2-\mu)(2+\eta-\lambda)\{4\gamma(1-\beta)(2-\alpha)+(1-\alpha)(1-\gamma+2\gamma\beta)\}}{8(2+\eta-\mu)(1+\gamma)(2-\alpha)} \tag{4.28}$$

The result is sharp with the extremal function defined by (4.26).

5- Further Distortion Properties

We next prove two further distortion theorems involving generalized fractional derivative operator $J_{0,Z}^{\lambda,\mu,\eta}$.

Theorem 5.1. *Let* $\lambda \geq 0;\ \mu < 2;\ \eta > max(\lambda,\mu) - 2$. *Also let the function* $f(z)$ *be defined by* (2.4) *be in the class* $T_{\lambda,\mu,\eta}^{*}(\alpha,\beta,\gamma)$. *Then*

$$\left|J_{0,Z}^{\lambda,\mu,\eta} f(z)\right| \geq \frac{\Gamma(2+\eta-\mu)}{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)}|z|^{1-\mu}\left\{1-\frac{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)}{(1+\gamma)(2-\alpha)}|z|\right\} \tag{5.1}$$

and

$$\left|J_{0,Z}^{\lambda,\mu,\eta} f(z)\right| \leq \frac{\Gamma(2+\eta-\mu)}{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)}|z|^{1-\mu}\left\{1+\frac{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)}{(1+\gamma)(2-\alpha)}|z|\right\} \tag{5.2}$$

for $z \in \mathcal{U}$. *The results* (5.1) *and* (5.2) *are sharp.*

Proof. Consider the function $P_{0,z}^{\lambda,\mu,\eta} f(z)$ defined by (2.11). With the aid of (4.7) and (4.14) we find that

$$\begin{aligned}\left|P_{0,z}^{\lambda,\mu,\eta} f(z)\right| &\geq |z| - \psi(2)|z|^2 \sum_{n=2}^{\infty} n a_n \\ &\geq |z| - \frac{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)}{(1+\gamma)(2-\alpha)}|z|^2\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\left|P_{0,z}^{\lambda,\mu,\eta} f(z)\right| &\leq |z| + \psi(2)|z|^2 \sum_{n=2}^{\infty} n a_n \\ &\leq |z| + \frac{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)}{(1+\gamma)(2-\alpha)}|z|^2\end{aligned} \tag{5.4}$$

which yields the inequality (5.1) and (5.2) of Theorem 5.1.

Finally, by taking the function $f(z)$ defined by

$$J_{0,Z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(2+\eta-\mu)}{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)} z^{1-\mu}\left\{1-\frac{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)}{(1+\gamma)(2-\alpha)} z\right\} \tag{5.5}$$

The results (5.1) and (5.2) are easily seen to be sharp. □

Corollary 5.2. *Let the function* $f(z)$ *defined by* (2.4) *be in the class* $T_{\lambda,\mu,\eta}^{*}(\alpha,\beta,\gamma)$. *Then* $J_{0,Z}^{\lambda,\mu,\eta} f(z)$ *is included in a disk with its centre at the origin and radius* r_3 *given by*

$$r_3 = \frac{\Gamma(2+\eta-\mu)}{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)}\left\{1+\frac{2\gamma(1-\beta)+(1-\alpha)(1+\gamma)}{(1+\gamma)(2-\alpha)}\right\} \tag{5.6}$$

Similarly we can establish the following result:

Theorem 5.3. *Let* $\lambda \geq 0;\ \mu < 2;\ \eta > max(\lambda,\mu) - 2$, *and let the function* $f(z)$ *be defined by* (2.4) *be in the class* $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. *Then*

$$\left|J_{0,Z}^{\lambda,\mu,\eta} f(z)\right| \geq \frac{\Gamma(2+\eta-\mu)}{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)}|z|^{1-\mu}\left\{1-\frac{2\gamma(1-\beta)(2-\alpha)+(1-\alpha)(1-\gamma+2\gamma\beta)}{(1+\gamma)(2-\alpha)}|z|\right\} \tag{5.7}$$

and

$$\left|J_{0,Z}^{\lambda,\mu,\eta}f(z)\right| \le \frac{\Gamma(2+\eta-\mu)}{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)}|z|^{1-\mu}\left\{1+\frac{2\gamma(1-\beta)(2-\alpha)+(1-\alpha)(1-\gamma+2\gamma\beta)}{(1+\gamma)(2-\alpha)}|z|\right\} \qquad (5.8)$$

for $z \in \mathcal{U}$. The results (5.7) *and* (5.8) *are sharp.*

Corollary 5.4. *Let the function $f(z)$ defined by* (2.4) *be in the class $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. Then $J_{0,Z}^{\lambda,\mu,\eta}f(z)$is included in a disk with its centre at the origin and radius r_4 given by*

$$r_4 = \frac{\Gamma(2+\eta-\mu)}{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)}\left\{1+\frac{2\gamma(1-\beta)(2-\alpha)+(1-\alpha)(1-\gamma+2\gamma\beta)}{(1+\gamma)(2-\alpha)}\right\} \qquad (5.9)$$

Remark 3. Letting $\mu=\lambda$ and using the relationship (2.9) in Theorem 5.1, Corollary 5.2, Theorem 5.3, and Corollary 5.4, we obtain the results which were proved by [Srivastava and Owa [11], Theorem 5, Corollary 3, Theorem 6, and Corollary 4, respectively].

6- Convexity Of Functions

In view of Lemma 2.4, we know that the function $f(z)$ defined by (2.4) is starlike in the unit disk $\mathcal{U}$ if and only if

$$\sum_{n=2}^{\infty} na_n \le 1 \qquad (6.1)$$

for $f(z) \in T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$, we find from (3.5) and (4.9) that

$$\sum_{n=2}^{\infty} na_n \le 2A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma) \le 1 \qquad (6.2)$$

where $A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ is defined by (4.6). Furthermore, for $f(z) \in C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$, we have

$$\sum_{n=2}^{\infty} na_n \le 2B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma) \le 1 \qquad (6.3)$$

where $B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ is defined by (4.24). Thus we observe that $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ are subclasses of starlike functions. Naturally, therefore, we are interested in finding the radii of convexity for functions in $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. We first state:

Theorem 6.1. *Let the function$f(z)$defined by* (2.4) *be in the class $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. Then $f(z)$is convex in the disk $|z| < r_5$, where*

$$r_5 = \inf_{n\ge 2}\left\{\frac{1}{2n\,A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)}\right\}^{1/(n-1)} \qquad (6.4)$$

and $A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ is given by (4.6). *The result is sharp.*

Proof. It suffices to prove

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \quad , \quad |z| < r_5 \qquad (6.5)$$

Indeed we have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right|$$
$$\leq \frac{\sum_{n=2}^{\infty} n(n-1)\, a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{n-1}} \tag{6.6}$$

Hence (6.5) is true if

$$\sum_{n=2}^{\infty} n(n-1)\, a_n |z|^{n-1} \leq 1 - \sum_{n=2}^{\infty} na_n |z|^{n-1} \tag{6.7}$$

that is, if

$$\sum_{n=2}^{\infty} n^2\, a_n |z|^{n-1} \leq 1 \tag{6.8}$$

with the aid of (4.14), (6.8) is true if

$$n|z|^{n-1} \leq \frac{1}{2A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)} \quad , \quad n \geq 2 \tag{6.9}$$

Solving (6.9) for $|z|$, we get

$$|z| \leq \left\{\frac{1}{2n\, A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)}\right\}^{1/(n-1)} \quad , \quad n \geq 2 \tag{6.10}$$

Finally, since $n^{-1/(n-1)}$ is an increasing function for integers $n \geq 2$, we have (6.5) for $|z| < r_5$, where r_5 is given by (6.4).

In order to complete the proof of Theorem 6.1, we note that the result is sharp for the function $f(z) \in T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ of the form

$$f(z) = z - \frac{2A_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)}{n} z^n \quad , \quad n \geq 2 \tag{6.11}$$

□

Similarly, we can prove the next theorem.

Theorem 6.2. *Let the function* $f(z)$ *defined by* (2.4) *be in the class* $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$. *Then* $f(z)$ *is convex in the disk* $|z| < r_6$, *where*

$$r_6 = \inf_{n\geq 2}\left\{\frac{1}{2n\, B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)}\right\}^{1/(n-1)} \tag{6.12}$$

and $B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ *is given by* (4.24). *The result is sharp for the function* $f(z) \in C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ *of the form*

$$f(z) = z - \frac{2B_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)}{n} z^n \,, \quad n \geq 2 \tag{6.13}$$

7- Conclusion

We have studied new classes $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ of univalent functions with negative coefficients defined by a certain fractional derivative operator in the unit disk $\mathcal{U}$. We obtained the sufficient conditions for the function $f(z)$ to be in $T^*_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(\alpha,\beta,\gamma)$.In addition, we derived a number of distortion theorems of functions belonging to these classes as well as distortion theorems for a certain fractional derivative operator of functions in theclasses. Also, we have determined the radii of convexity for functions belonging to these classes.

Some of the known results follow as particular cases from our results; see for example, Gupta [1]; Srivastava and Owa[10] and Srivastava and Owa [11].

References

[1] Gupta, V. P. (1984): convex class of starlike functions, Yokohama Math. J. **32**, pp. 55-59.

[2] Owa, S. (1978): On the distortion theorems. I, Kyungpook Math. J. **18**, pp. 53-59.

[3] Owa, S. ; Srivastava, H. M. (1987): Univalent and starlike generalized hypergeometric functions, Canad. J. Math. **39**, pp. 1057-1077.

[4] Raina, R. K. ; Srivastava, H. M. (1996): A certain subclass of analytic functions associated with operators of fractional calculus, Comput. Math. Appl. **32**, pp. 13-19.

[5] Robertson, M. S. (1936): On the theory of univalent functions, Ann. of Math. **37**, pp. 374-408.

[6] Silverman, H. (1975):Univalent functions with negative coefficients, Proc. Amer. Math. Soc. **51**, pp. 109-116.

[7] Srivastava, H. M. ; Karlsson, P. W. (1985): Multiple Gaussian hypergeometric series, Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York/Chichester/Brisbane/Toronto.

[8] Srivastava, H. M. ; Owa, S. (Eds) (1989): Univalent functions, fractional calculus, and their applications, Halsted Press/Ellis Horwood Limited/John Wiley and Sons, New York/Chichester/Brisbane/Toronto.

[9] Srivastava, H. M. ; Owa, S. (Eds) (1992): Current topics in analytic function theory, World Scientific Publishing Company, Singapore, New Jersey London and Hong Kong.

[10] Srivastava, H. M. ; Owa, S. (1991): Certain subclasses of starlike functions - I, J. Math. Anal. Appl. **161**, pp. 405-415.

[11] Srivastava, H. M. ; Owa, S. (1991): Certain subclasses of starlike functions - II, J. Math. Anal. Appl. **161**, pp. 416-425.

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An Application Of Inner Product Spaces

" Conditioning And the Rayleigh Quotient "

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الملخص:

في هذه الورقة سندرس تطبيق عملي لفضاء الضرب الداخلي , وكيف يستخدم لحل مجموعة من المعادلات على الصورة :

AX=b

حيث A مصفوفة من الرتبة m×n , b متجه من الرتبة m×1 .

Abstract:

In this paper, we introduce an application on inner product spaces and how used to solve a system of linear equations in the form Ax=b , where A is m×n matrix and b is m×1 vector.

Introduction:

The system of linear equations in the form Ax=b, where A is an m × n matrix and b is an m × 1 vector often arise in applications to the real world. The coefficients in the system are frequently obtained from experimental data, and in many cases, both m and n are so large that a computer must be used in the calculation of the solution. Thus two types of errors must be considered. First , experimental errors arise in the collection of data since no instruments can provide completely accurate measurements. Second, computers introduce roundoff errors. One might intuitively feel that small relative changes in the coefficients of the system cause small relative errors in the solution. A system that has this property is called well-conditioned ,otherwise, the system is called ill conditioned We now consider some examples of these types of errors, concentrating primarily on changes in b rather that on changes in the entries of A . In addition ,we assume that A is a square , complex(or real), invertible matrix since this is the case most frequently encountered in applications.

Definition 1 :(4)

A set **V** is called a vector space over a field **F** if :-

(a) under binary operation called addition (+)

(i) V is closed

(ii) u + v = v + u for all u, v $\in$ V (commutative axiom)

(iii) u + (v + w) = (u + v) + w for all u, v, w $\in$ V

(associative axiom)

(iv) there exists an element O $\in$ V, called a zero element such that u + O = O + u for all u $\in$ V.

(v) for every v $\in$ V, there exists an element (-v) $\in$ V called an inverse of v such that v + (-v) = 0 = (- v) + (v).

(b) under the scalar multiplication

(i) V is closed : α u $\in$ V $\forall$ $\alpha \in$ F , $\forall$ u $\in$ V.

(ii) α (u + v) = α u + α v for all u, v $\in$ V ; $\alpha \in$ F.

(iii) ($\alpha + \beta$) v = α v + β v for all v $\in$ V ; $\alpha , \beta \in$ F.

(iv) α (β v) = (α β) v for all v $\in$ V ; $\alpha , \beta \in$ F.

(v) 1 v= v for all v $\in$ V ; 1 $\in$ F.

Definition 2 : (4)

Let V be a vector space over F . An inner product on vector space V is an operation that assigns to every pair of vectors u and v in V a scalar <u,v> in F such that the following properties hold for all vectors u , v and w in V and all scalars c in F.

(1) $\overline{< u,v >} = < v , u >$, where the bar denotes complex conjugation
(2) $< u + v , w > = < u , w > + < v , w >$
(3) $< cu , v > = c < u , v >$
(4) $< u , u > \geq 0$, and $< u , u > = 0$ if and only if u = 0 .

Now , we will introduce some examples to apply this application as following :

Example 1

Consider the system

$$x_1 + x_2 = 5$$

$$x_1 - x_2 = 1$$

The solution to this system is

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Now suppose that we change the system somewhat and consider the new system

$$x_1 + x_2 = 5$$

$$x_1 - x_2 = 1.0001$$

This modified system has the solution

$$\begin{bmatrix} 3.00005 \\ 1.99995 \end{bmatrix}.$$

We see that a change of 10^{-4} in one coefficient has caused a change of less than 10^{-4} in each coordinate of the new solution. More generally, the system

$$x_1 + x_2 = 5$$

$$x_1 - x_2 = 1+h$$

has the solution

$$\begin{bmatrix} 3 + \frac{h}{2} \\ 2 - \frac{h}{2} \end{bmatrix}.$$

Hence small changes in b introduce small changes in the solution . Of course, we are really interested in relative changes since a change in the solution of , say , 10. is considered large if the original solution is of the order 10^{-2} , but small if the original solution is of the order 10^{6} .

We use the notation δb to denote the vector $b' - b$, where b is the vector in the original system and b' is the vector in the modified system . Thus we have :

$$\delta \mathrm{b} = \begin{bmatrix} 5 \\ 1+\mathrm{h} \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathrm{h} \end{bmatrix}$$

We now define the relative change in b to be the scalar $\|\delta b\| / \|b\|$, where $\|.\|$ denotes the standard norm on C^n or R^n ; that is , $\|b\| = \sqrt{<b,b>}.$ Most of what follows, however, is true for any norm. Similar definitions hold for the relative change in x.

In this example :

$$\frac{\|\delta b\|}{\|b\|}=\frac{|h|}{\sqrt{26}}$$ and

$$\frac{\|\delta x\|}{\|x\|}=\frac{\left\|\begin{bmatrix}3+(h/2)\\2-(h/2)\end{bmatrix}-\begin{bmatrix}3\\2\end{bmatrix}\right\|}{\left\|\begin{bmatrix}3\\2\end{bmatrix}\right\|}=\frac{|h|}{\sqrt{26}}$$

Thus the relative change in x equals, coincidentally, the relative change in b; so the system is well-conditioned.

Example 2

Consider the system

$$x_1 + x_2 = 3$$

$$x_1 + 1.00001\, x_2 = 3.00001\ ,$$

which has $\begin{bmatrix}2\\1\end{bmatrix}$

as its solution . The solution to the related system

$$x_1 + x_2 = 3$$

$$x_1 + 1.00001\, x_2 = 3.00001 + h$$

is

$$\begin{bmatrix}2-(10^5)h\\1+(10^5)h\end{bmatrix}$$

Hence,

$$\frac{\|\delta x\|}{\|x\|}=\frac{\left\|\begin{bmatrix}2-(10^5)h\\1+(10^5)h\end{bmatrix}-\begin{bmatrix}2\\1\end{bmatrix}\right\|}{\left\|\begin{bmatrix}2\\1\end{bmatrix}\right\|}$$

$$=10^5\sqrt{2/5}\,|h| \quad \geq \quad 10^4\,|h|,$$

While

$$\frac{\|\delta b\|}{\|b\|} \cong \frac{\|h\|}{3\sqrt{2}}$$

Thus the relative change in x is at least 10^4 times the relative change in b! This system is very ill-conditioned . Observe that the lines defined by the two equations are nearly coincident. So a small change in either line could greatly alter the point of intersection, that is , the solution to the system.

To apply the full strength of the theory of self-adjoint matrices to the study of conditioning, we need the notion of the norm of a matrix .

Definition 3 : (2)

An n × n real or complex matrix A is self-adjoint (Hermitian) if $A = A^*$

(i.e., such that $(A)_{ij} = \overline{A_{ji}}$ for all i , j) .

Definition 4 : (3)

Let A be a complex (0r real) n × n matrix. Define the Euclidean norm of A by

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

where $x \in C^n$ or $x \in R^n$.

Intuitively, $\|A\|$ represents the maximum magnification of a vector by the matrix A. The question of whether or not this maximum exists, as well as the problem of how to compute it, can be answered by the use of the so-colled Rayleigh quotient.

Definition 5 : (2)

Let B be an n × n self- adjoint matrix. The Rayleigh quotient for $x \neq 0$ is defined to be the scalar $R(x) = < Bx.x > / \|x\|^2$.

The following result characterizes the extreme values of the Rayleigh quotient of a self-adjoint matrix.

Theorem 1 : (1)

For a self-adjoint matrix $B \in M_{nxn}(F)$, we have that $\max_{x \neq 0} R(x)$ is the largest eigenvalue of B and $\min_{x \neq 0} R(x)$ is the smallest eigenvalue of B.

Remark 1

Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is called an eigenvector of T if there exist a scalar λ such that $T(v) = \lambda v$. The scalar λ is called eigenvalue corresponding to the eigenvector v.

Corollary 1: (1)

For any square matrix A, $\|A\|$ is finite and , in fact, equals $\sqrt{\lambda}$ where λ is the largest eigenvalue of A*A.

Lemma 1 : (1)

For any square matrix A, λ is an eigenvalue of A*A if and only if λ is an eigenvalue of AA*.

Corollary 2 : (1)

Let A be an invertible matrix . Then $\|A^{-1}\| = 1/\sqrt{\lambda}$, where λ is the smallest eigenvalue of A*A.

For many applications, it is only the largest and smallest eigenvalues that are of interest.

For example, in the case of vibration problems, the smallest eigenvalue represents the lowest frequency at which vibrations can occur.

We see the role of both of these eigenvalues in our study of conditioning.

Example 3

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Then

$$B = A*A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

The eigen values of B are found as follows :

We solve the equation:

Det (B- tI_3) = 0

$$B - tI_3 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix}$$

$$= \begin{bmatrix} 2-t & -1 & 1 \\ -1 & 2-t & 1 \\ -1 & 1 & 2-t \end{bmatrix}$$

$$\det(B - tI_3) = \begin{vmatrix} 2-t & -1 & 1 \\ -1 & 2-t & 1 \\ 1 & 1 & 2-t \end{vmatrix}$$

= (2-t) (4-4t+t^2) – 6+3t-2

= 8 – 8t + 2t^2 – 4t + 4t^2 –t^3 -6 + 3t -2

= -t^3+ 6t^2 – 9t

= t [-t^2+ 6t – 9]

Det(B- tI_3) = 0 , then

t[t^2-6t + 9] = 0

Hence,

t = 0 , t^2 – 6 t + 9 = 0

t= 0 (t-3 $)^2$ = 0

The eigen values of B are 3 , 3 and 0

Therefore

$$\|A\| = \max_{x \neq 0} \frac{\|AX\|}{\|x\|} \text{ for any } x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq 0$$

Then , by corollary $\|A\| = \sqrt{3}$

We may compute R(x) for the matrix B as $R(x) = \frac{<Bx, x>}{\|x\|^2}$

$$Bx = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 2a - b + c \\ -a + 2b + c \\ a + b + 2c \end{bmatrix}$$

$$<Bx, x> = a(2a - b + c) + b(-a + 2b + c) + c(a + b + 2c)$$

$$= 2a^2 - ab + ac - ba \_ ab^2 + bc + ca + cb + 2c^2$$

$$= 2a^2 + 2b^2 + 2c^2 + 2ac + 2bc - 2ba$$

$$\|x\|^2 = a^2 + b^2 + c^2$$

$$R(x) = \frac{<Bx, x>}{\|x\|^2} = \frac{2(a^2 + b^2 + c^2 - ab + ac + bc)}{a^2 + b^2 + c^2} \le 3$$

Definition 6 : (1)

The number $\|A\|.\|A^{-1}\|$ is called the condition number of A and is denoted cond (A).

Theorem 2 : (1)

For the system Ax= b , where A is invertible and $b \neq 0$ the following statement are true .

(a) For any norm $\|.\|$, we have

$$\frac{1}{cond\,(A)}\frac{\|\delta b\|}{\|b\|}\le\frac{\|\delta x\|}{\|x\|}\le cond\,(A)\frac{\|\delta b\|}{\|b\|}$$

(b) If $\|.\|$ is the Euclidean norm, then cond $(A)=\sqrt{\lambda_1/\lambda_n}$, where λ_1 and λ_n are the largest and smallest eigenvalues , respectively , of A^*A .

Proof:

For a given δb , let δx are the vector that satisfies

$A(x+\delta x)=b+\delta b$

$Ax+A(\delta x)=b+\delta b$

$A(\delta x)=\delta b$ (1)

$A^{-1}A(\delta x)=A^{-1}(\delta b)$

$\delta x=A^{-1}(\delta b)$ (2)

a) (i) $\|b\|=\|Ax\|\le\|A\|\|x\|$, so we get

$$\frac{1}{\|b\|}\ge\frac{1}{\|A\|\,\|x\|}$$

$$\frac{1}{\|x\|}\le\frac{\|A\|}{\|b\|}$$

Now , from (2) we get

$$\|\delta x\|=\|A^{-1}(\delta b)\|\le\|A^{-1}\|\,\|\delta b\|$$

$$\frac{\|\delta x\|}{\|x\|}\le\frac{\|A\|}{\|b\|}\,.\|A^{-1}\|\,\|\delta b\|=\|A\|\,\|A^{-1}\|\,\frac{\|\delta b\|}{\|b\|}=cond\,(A)\,\frac{\|\delta b\|}{\|b\|}$$

(ii) x = Ab , then

$$\|x\|=\|A^{-1}\|\|A^{-1}b\|\le\|A^{-1}\|\,\|b\|$$

Hence ,

$$\frac{1}{\|x\|} \geq \frac{1}{\|A^{-1}\| \, \|b\|}$$

$$\frac{1}{\|A^{-1}\|\|b\|} \leq \frac{1}{\|x\|}$$

Now , from (1) , we get

$$\|\delta b\| \leq \|A\| \, \|\delta b\|$$

Hence ,

$$\frac{\|\delta b\|}{\|A^{-1}\| \, \|b\|} \leq \frac{\|A\| \, \|\delta x\|}{\|x\|}$$

So we get

$$\frac{1}{\|A\|\|A^{-1}\|} \frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|}$$

Hence ,

$$\frac{1}{\text{cond }(A)} \frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|}$$

(b) From corollary 3.1 and corollary we have $\|A\| = \sqrt{\lambda_1}$, $\|A^{-1}\| = \frac{1}{\sqrt{\lambda_n}}$ where λ_1 and λ_n are the smallest and largest eigenvalues, respectively of A*A .

Then $cond\,(A) = \|A\| . \|A^{-1}\| = \sqrt{\frac{\lambda_1}{\lambda_n}}$.

Conclusion :

The solution of linear equations is of great importance in linear algebraic science .So we used Rayleigh Quotient to solve the linear equations of the form Ax = b , where A is a m×n matrix and b is a m × 1 vector .

References :-

(1) – Friedberg , s , H , Insel , A, J and Spence , l , E - " Linear Algebra " Pearson Education, Inc (2003). Illinois State University .

(2) Morris, A, O - "Linear algebra An introduction**" - (1982) -** Vnr New Mathematics Library9 .

(3) Lay David C " Linear Algebra and Its Applications " **– (2006) -** university of mary land **_** college park.

(4) Bretscher, O - "Linear Algebra with Applications" - Pearson Education Inc (2005) - Colby College.

WHY THE ARAB LEARNERS OF ENGLISH LANGUAGE COMMITT ERRORS IN PRONUNCIATION AND GRAMMAR : A PSYCHOLINGUISTIC APPROACH TO THE STUDY

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Abstract

This paper makes a psycholinguistic approach to observe and analyze why the Arab learners of English language commit errors in pronunciation and grammar. It also presents the suggestions.

First, it phonologically analyzes the English phonotactics in the English of Arab learners of English as a foreign language to determine the types of pronunciation difficulties they encounter. More specifically, it investigates the types of declusterization processes found in their interlingual communication and the sources of such processes. The results of this study demonstrate that Arab learners of English unintentionally insert an anapestic vowel in the onset as well as in the coda of certain English syllables. Results also show that the major reason for declusterization processes is the mother tongue influence.

In order to overcome such difficulties, this paper suggests a new approach for teaching and learning English language syllable structure system.

Secondly, it focuses on the errors committed in grammar by analyzing it psycholinguistically.

Objectives And Methodology

This study aims at:

1. Identifying ,classifying, and analyzing errors of insertion made by Arab learners of English in the area of pronunciation,
2. Finding out the possible sources of these errors, and
3. Suggesting teaching procedures that help teachers and students overcome the areas of difficulty.

-**Methodology**:

This study follows a psycholinguistic approach to analyse the errors and mistakes committed by Arab learners of English language in pronunciation and grammar.

- Participants of the study :

This study is conducted by observing the performance of B.A. Ist and IInd year English language learners in the College of Arts and Sciences Kasr Al-Khiyar ,Al-Margib University . These students were scrutinized for committing errors in pronunciation and grammar specific to Arab students .

INTRODUCTION

The ultimate goal of most foreign language learners is to attain native like fluency. They want to be indistinguishable from native speakers. However, for many learners, this dream has remained a dream and has not come true especially in the area of pronunciation as native speakers usually identify them as non-native speakers because of their accent. A large number of foreign language learners believe that the main difficulty they encounter when speaking the foreign language is pronunciation and consider this difficulty as the main source for their communication problems (Altaha, F 1995: 34).

English occupies a high status among world international languages, as it has become the language of diplomacy, trade, communication, technology and business. Thus, learning English provides the person with an advantage as an active participant in today's world, opening new horizons to a better future (Cook, V.J. 1992: 140).

English as an international language has been taught in almost all countries in the world. In Libya English is a foreign language which is a compulsory subject to be taught in all schools from lower secondary to upper secondary schools to under-graduate students in universities ;even in elementary schools. However, we have seen that the proficiency in English of secondary school graduates still creates disappointment among teachers themselves as well as parents. The unsatisfying quality of English in Arab countries of course is related to different variables.

we have tried to shed light on some of these variables here .That is to say, the causes behind the errors committed in pronunciation and grammar in English language by the Arab learners.

The importance of investigating pronunciation and grammar difficulties stems from the fact that, it stands as an obstacle in communication. However, it is necessary, in this research, to find out why the aforesaid learners face difficulty in the acquisition of the phonological system and grammatical structure of any non-native language (English).

1.ERRORS IN PRONUNCIATION AND ITS CAUSES

Errors in pronunciation of any non-native speaker of any language is mostly impeded by the influence of mother tongue .However, the Arabic and English phonological systems vary extensively, not only in the range of sounds used, but also in the relative importance of vowels and consonants in expressing meaning. While English has 22 vowels and diphthongs to 24 consonants, Arabic has only eight vowels and diphthongs to 32 consonants.

أ	A	ض	Dd
ب	B	ط	Tt
ت	T	ظ	Dhh
ث	Th	ع	'A
ج	J	غ	R
ح	H~	ف	F
خ	Kh	ق	Q'
د	D	ك	K
ذ	Dh	ل	L
ر	R	م	M
ز	Z	ن	N
س	S	هـ	H
ش	Ch	و	Ou
ص	Ss	ي	Y

1.1 Consonant Clusters

English has far more consonant clusters than Arabic.Some initial two-segment clusters which Arabic does not have corresponding equivalents to, include: pr, pl, gr, thr, thw, sp. The three-segment initial consonant clusters are entirely absent in Arabic, e.g., clusters such as spr, skr, str, spl. Faced with the challenge of such consonant clusters, Arabic speakers often insert short vowels in order to "assist" pronunciation in the following manner:

'perice'	**or**	***'pirice'***	**for**	***price***
'ispring'	**or**	***'sipring'***	**for**	***spring***

The range of consonant clusters appearing at the end of words is also much smaller in Arabic. In dramatic contrast to English, which has 78 three-segment clusters and fourteen four-segment clusters occurring at the end of words, Arabic has none. Again, faced with such terminal clusters, Arabic speakers tend to insert short vowels to assist pronunciation:

'arrangid'	**for**	***arranged***
'monthiz'	**for**	***months***
'nexist'	**for**	***next***
'sikas'	**for**	***six***
'lookas'	**for**	***looks***

1.2 Insertion of /ɪ/ in the onset

In all the following English monosyllabic words, the onset consists of three consonants; actually, such combinations pose difficulties for Arab learners of English as their native dialect does not allow clusters of the type CCC initially. As a result, they insert the high front short vowel /ɪ/ which declusterizes the clusters to ease their pronunciation. What can be inferred here is that insertion is a rule governed process as all participants insert the above vowel after the first member of the consonant cluster.

1. ***/sɪblaʃ/ splash***
2. ***/sɪblɪ:n/ spleen***
3. ***/sɪkrɪ:n/ screen***
4. ***/sɪbraɪt/ sprite***
5. ***/sɪtreɪn/ strain***
6. ***/sɪkrap/ scrap***
7. ***/sɪtreɪt/ straight***
8. ***/sɪpreɪ/ spray***

Teachers often encounter examples of such pronunciations, which also can carry over into the spelling of such English words by students whose mother tongue is Arabic.

1.3 Influence of English Spelling on Pronunciation

While there are no similarities between the Arabic and English writing systems, Arabic spelling within its own system is simple and virtually phonetic. Letters stand directly for their sounds. Arabic speakers attempt, therefore, to pronounce English words using the same phonetic methodology. Add to this the salience of consonants in Arabic and you get severe pronunciation problems caused by the influence of the written form:

'istobbid' for stopped (the 'p' sound does not exist in Arabic)
'forigen' for foreign.

1.4 Rhythm and Stress

Arabic speakers can have problems grasping the unpredictable nature of English word stress since Arabic is a stress-timed language. In stark contrast with English, word stress in Arabic is predictable and regular. The idea that stress can alter meaning, as in con'vict (verb) and 'convict (noun) is utterly foreign. Arabic words that are spelled identically often appear, and mean completely different things, but will have dissimilar short vowels which count as sounds and change the meaning altogether.

Phrase and sentence rhythms are similar in both Arabic and English languages, and cause few problems. Primary stresses occur more frequently in Arabic while unstressed syllables are pronounced more clearly. As with English, the unstressed syllable has neutral vowels, but such vowels are not 'swallowed' as in English. Arabs reading English often avoid contracted forms and elisions, and read with a rather heavy staccato rhythm.

1.5 Intonation

Intonation patterns in Arabic are similar to English in contour and meaning. However, Arabic speakers use rising tones rather than structural markers to denote questions, suggestions and offers far more frequently than English-speakers, and this practice is often carried over into the spoken English of Arabic speakers.

When reading aloud however, as opposed to talking, the Arabic speaker tends to intone or chant, reducing intonation to a low fall at the ends of phrases and sentences. Speech making, news reading and religious recitation are all quite different in rhythm and intonation from normal speech. Consequently, Arabic speakers called on to read aloud in front of a group may produce a very unnatural recitation because they see the process of formal reading as distinct from everyday speech.

In their attempt to identify problems that encounter Arab students of English at initial stages, Kharma & Hajjaj (1989) present four major areas of difficulty. As far as consonants are concerned, they presented two problematic issues. First, certain pairs are confused by learners such as /ʧ/ and /ʃ/ as in chair and share ; /v/ and /f/ as in fast and vast; /dʒ/ and /ʒ/ as in /dʒɑ:/ jar and /ʒɑ:/ jar; /p/ and /b/ as in pin and bin; /ŋ/ and /n/ as in /sɪŋ/ sing and /sɪng/ sing; /s/ and /θ/ as in sin and thin. Second, learners insert a short vowel to break down the long consonant clusters to pronounce them as in /sɪprɪŋ/ for spring; /wɪʃɪd/ for wished; /ɑ:skɪd/ for asked (Kharma & Hajjaj, 1989: 14). In vowels, two types of difficulty are identified. First, certain diphthongs are replaced by other sounds due to L1 interference for example, /eə/ →/eɪ/; /ʊə/ → /u:/;/ɪə/ → /ɪ:/;

and /əʊ/→ /ɔ:/. Second, the distinction between certain pairs of vowels as in /ɪ/ and /e/ as in sit and set; /ʌ/ and /ɒ/ as in luck and lock; /əʊ/ and /ɔ:/ as in coat and caught (Kharma & Hajjaj, 1989, p. 16).

Analyzing the pronunciation errors experienced by five Saudi learners of English as a second language, Binturki (2008) investigates the difficulties in producing the voiceless bilabial stop /p/, the voiced labiodental fricative /v/ , and the alveolar approximant // especially what word environments are most difficult for participants. His results show that participants have difficulty with the three-targeted consonants, but the greatest is with /v/. The study also finds that difficulty is closely related to certain word positions, so all the three sounds are used more accurately when occurring in word initial position than in word final position.

Tushyeh (1996) investigates errors committed by Arab learners of English at various linguistic levels. At the phonological level, participants have a difficulty in distinguishing the following pairs: /p/ and /b/, /f/ and /v/, and /ɪ/ and /e/.

Wahba (1998) focuses his study on problems encountered by Egyptian learners of English as a second language and concludes that certain phonological errors made are related to stress and intonation. These errors are interlingual ones; attributed to phonological differences between the sound systems of English and Arabic.

In order to see the influence of ones L1 on the acquisition of the L2 pronunciation, Barros (2003) identifies and analyzes the difficulties encountered by Arabic speakers when pronouncing English consonants. The participants were a group of Arabic speakers came from different Arab countries with different colloquial Arabic backgrounds. All participants were in contact with the target language group and culture after the age of puberty for at least four years. The results show that eight English consonants, namely, /ŋ/, /p/, /v/, /d/, /l/, /ʤ/, //, and /r/ are identified as problematic ones for Arabic speakers. The author also finds that interference of L1 seems to be the major factor contributing to pronunciation problems that might differ from one Arabic speaker to another, depending on the colloquial variety of Arabic they use.

1.6 Syllable Structures in Modern Standard Arabic(MSA):

It is necessary to have a quick look at the syllable structures in Modern Standard Arabic (MSA) and in English language.

In MSA, the syllable structure may be expressed by the following formula: CV(V)(C)(C). Therefore, the following syllable types are admissible:

a. CV

b. CVV

c. CVC

d. CVVC

e. CVCC

f. CVVCC

There is some difference between MSA syllable structure and that of the participants(Jordan) Ammani dialect of Arabic; for example, the syllable CVVCC does not exist in Ammani Arabic while CVCC is not a common one. Another syllable structure, namely, CCVC is found in Ammani Arabic but not in MSA.

English syllable may be expressed by the formula: (C)(C)(C)V(C)(C)(C)(C). The following syllables exist in English:

a. V

b. CV

c. VC

d. CVC

e. CCV

f. VCC

g. CCVC

h. CCVCC

i. CCCV

j. CCCVCC

k. CCCVCCC

l. CVCCCC

The errors found in this study fall under three types namely, (i) insertion, (ii) substitution and (iii) deletion. As far as the declusterization process is concerned, attention is paid only to the insertion type .Therefore, substitution and deletion types are not tackled in this study. As mentioned above, learners native language interference is indispensible.

As evident from the above syllable structures, the systems are different. Many English syllables are predicted to be difficult for Arab learners since they do not exist in Arabic language. In Arabic language, onset is an obligatory element in the structure of any syllable and it should be always C which means that no word is allowed to begin with a vowel sound. In other words, no two consonants are allowed to meet in the beginning of any word without being separated by a

vowel. The coda of the syllable is optional in the above structures since some syllable types are open (i. e. ending in a vowel). So the coda can be zero, one or two consonants but not more.

Hence these given causes are enough to justify their errors and difficulties to gain proficiency in English language pronunciation.

2-ERRORS IN GRAMMAR AND THEIR CAUSES

The grammatical structure of Arabic, a Semitic language, is very different from that of Indo-European languages such as English. These great differences must be borne in mind while teaching to the Arabic speakers.

The basis of the Arabic language is the three-consonant root. A notion such as writing, cooking, or eating is represented by three consonants in a particular order. All verb forms, nouns, adjectives, participles, etc. are then formed by putting these three-root consonants into fixed vowel patterns, modified sometimes by simple prefixes and suffixes.

Example #1

Root /k/ /t/ /b/ (=writing)
A person who does this for a living katib (= a writer)
Passive participle maktoob (= written)
Present tense yaktubu (= he writes)

Example #2

Root /j/ /r/ /h/ (= wounding or cutting)
A person who does this for a living is" jarraah" (= a surgeon)
Passive participle majrooh (= wounded or a battle casualty)
Present tense yajruhu (= he wounds him)

There are over 50 such patterns. While not all forms are found for each root, the three-consonant root is the structural basis of the language.

It follows that Arabic speakers have great difficulty in grasping the confusing range of patterns for all words in English; that nouns, verbs, and adjectives follow no regular patterns to distinguish one from another, and may, indeed, have the same orthographic form. Such regularities of morphology as English has, particularly, in the area of affixes, will be readily grasped by Arabic speakers, e.g. -ing, -able, un-, etc.

2.1 Word Order

In formal written Arabic, the verb comes first followed by the subject. This convention is followed more in writing than in speech, and may transpose to English writing:

e.g. Decided the minister yesterday to visit the school.

2.2 Questions and Negatives; Auxiliaries

The auxiliary "do" has no equivalent in Arabic. Where no specific question word is used, a question is marked only by its rising intonation:

e.g. ***-You went to London?***

-You like coffee?

Note that the Arabic for "where?" is ("Ayna " which is often confused with dialect "wayn?", "when".

Negatives are formed by putting a particle (laa or maa) before the verb:

e.g. He not play football.

2.3 The Verb to Be

There is no verb "to be" in Arabic in the present tense. The copula (am, is, are) is not expressed. It is therefore, commonly omitted in English by Arabic speakers, particularly in present progressive forms:

e.g. He teacher.
The boy tall.
He going to school.

2.4 Pronouns

Arabic verb forms incorporate the personal pronouns, subject and object, as prefixes and suffixes. It is common to have them repeated in English as part of the verb:

e.g. John he works there.

2.5 Articles

There is no indefinite article in Arabic, and the definite article has a range of use different from English. The indefinite article causes particular problems as it is commonly omitted with singular and plural countables:

e.g. - This is book or This book (for - This is a book)

- He was soldier

When the English indefinite article has been learned by the Arabic speaker, it tends to be used wherever the definite article is not used:

e.g. - There are a books.

- I want a rice.

There is a definite article form in Arabic, though it takes the form of a prefix (al-). It is used, as in English, to refer back to indefinite nouns previously mentioned, and also for unique reference (the sun, on the floor, etc.)

The most common problem with the definite article arises from interference from the Arabic genitive construction:

English	***Arabic***
John's book. (or The book of John.)	***Book John.***
A man's work. (or The work of a man.)	***Work man.***
The teacher's car. (or The car of the teacher.)	***Car the teacher.***

Most errors of word order and use of articles in genitive constructions are interference of this kind:

e.g.* - *This is the book the teacher.

- This is the key door.

It follows that Arabic speakers have great difficulties with the Saxon genitive construction.

The special cases, in which English omits the article, e.g. in bed, at dawn, on Thursday, for breakfast, etc. usually take the definite article in Arabic:

e.g.* - *At the sunset we broke our fast.

- What would you like for the breakfast?

All days of the week, some months in the Muslim calendar, and many names of towns, cities and countries include the definite article in Arabic, which is often translated, appropriately or not:

e.g.* - *We lived in the Cairo.

- We had a flat in the Khartoum.

- On Monday we went to Cardiff.

2.6 Adjectives and Adverbs

Adjectives follow nouns in Arabic and agree in gender and number. This may cause beginners to make mistakes:

e.g.* - *He is man tall. (for He is a tall man.)

Adverbs are used less commonly in Arabic than in English and, except for adverbs of time; do not have a fixed pattern. Adverbs of manner are often expressed in a phrase: quickly is expressed "with speed", and dangerously as "in a dangerous way." There is frequent confusion between the adjective and adverb forms in English, and the adjective form is usually overused:

e.g.* - *He drives very dangerous.

2.7 Prepositions and Particles

Arabic has a wealth of fixed prepositions and particles, with both verbs and adjectives. Many of these do not coincide with their direct English translations:

e.g. ***to arrive to***

to be short of

afraid from

angry on

near from

an expert by

Some prepositions have verbal force:

· "On" expresses obligation:

e.g. It is on me that I pay him.

"To" and "for" express possession:

e.g. This book is to me / for me. (for This book is mine.)

· "With" expresses present possession:
e.g. With me my camera. (for I have my camera with me.)

· "For" expresses purpose:
e.g. I went home for (I) get my book. (for I went home to get my book.)

2.8 The Active and Passive Voices

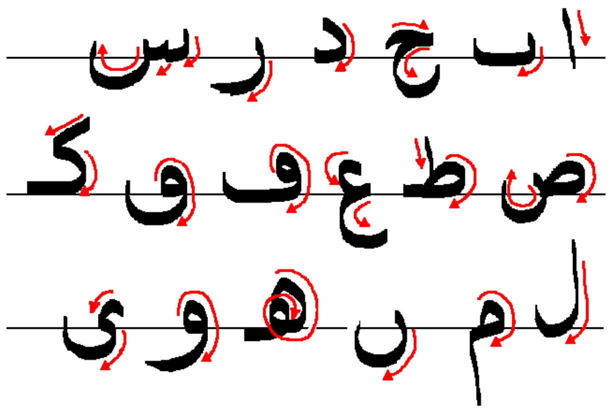
There are active and passive forms for all tenses in Arabic, but they are virtually identical, differing only in the (unwritten) short vowel. A passive verb in a text is therefore only recognizable as such from its context. The passive voice is used far less frequently used in Arabic writing than in English, and hardly at all in everyday speech. Thus while the concepts of active and passive will readily be understood, the uses and forms of the passive cause problems.

2.9 Vocabulary

The acquisition of vocabulary is particularly difficult for Arab learners of English. Only a minimal number of words in English are borrowed from Arabic. A small range of mainly technical words, such as computer, radar, helicopter, and television, have been taken into Arabic, but these are common to most languages. Arabic speakers have very few aids to reading and listening comprehension by virtue of their first language, and they should not be expected to acquire English at anything like the same pace as European learners.

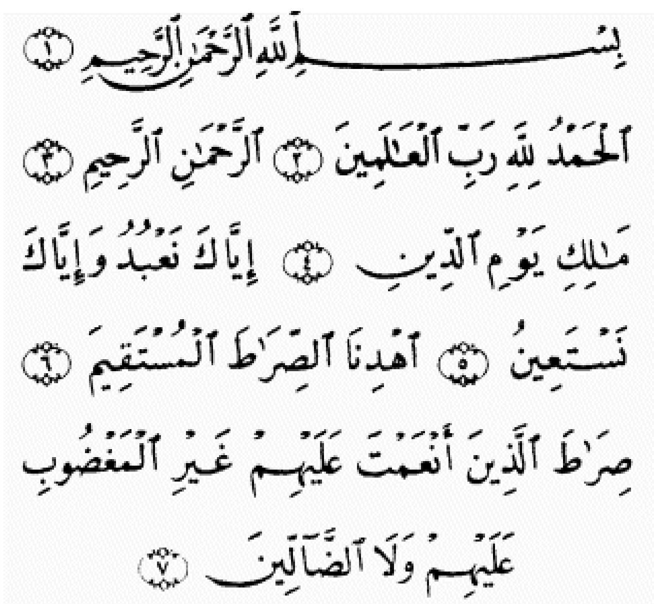
2.10 -Writing System (Orthography and Punctuation)

Arabic orthography is a cursive system, running from right to left. Only consonants and long vowels are written. There is no upper and lower case distinction, nor can the isolated forms of letters normally be juxtaposed to form words.



Arabic speakers must, hence, learn an entirely new alphabet for English, including a capital letter system; and then master its rather unconventional spelling patterns. All aspects of writing in English cause major problems for Arabic speakers, and they should not be expected to cope with reading or writing at the same level or pace as European students who are at a similar level of proficiency in oral English.

A Writing Sample



1. In the name of Allah, the Beneficent, the Merciful.
2. Praise be to Allah, the Lord of the Worlds.
3. The Beneficent, the Merciful.
4. Master of the Day of Judgment.
5. Thee (alone) do we worship, and Thee (alone) we ask for help.
6. Show us the straight path.
7. The path of those whom Thou hast favored; not (the path) of those who earn Thy anger nor of those who go astray.

Opening sura (chapter) of the Qur'an

2.11 Punctuation

Arabic punctuation is now similar to western style punctuation, though some of the symbols are inverted or reversed, e.g. a reversed question mark and comma. The use of full stops and commas is much freer than in English, and it is common to begin each new sentence with And or So. Connected writing in English tends therefore to contain long, loose sentences, linked by commas and "ands."

3.SOURCES OF DIFFICULTY

The source of any error in language learning can be overgeneralization, omission - as a learning strategy, spelling-to-sound rules, stage of development or learners mother tongue interference. What is relevant to this study is (i) interference and (ii) stage of development. They are discussed below.

a) Interference

Learners of any language, whether L1 or L2, form hypotheses about the rules of the language they are learning. In L2 situation, they sometimes rely on their L1 background to form such hypotheses that will result in a successful or erroneous structure, depending on the feature or rule being transferred. As far as the English syllable structure is concerned, it is clear that certain English syllable types do not exist in Arabic and they pose difficulties for Arab learners in different ways. When looking at the structure of the English permitted onsets, one finds that the combinations: CC and CCC are going to be problematic ones for Arab learners of English in general. CC- does not pose any difficulty for these learners in particular as it is used in their colloquial variety of Arabic. English permitted codas are more problematic ones than onsets as the number of consonant members is relatively high. The following combinations are predicted to form difficulties for learners: CCC and CCCC. It is believed that vowels drag words, that is to say, without vowels it is difficult to produce a string of consonants, as it is difficult for any speaker to move from one place of articulation to another where the articulators are very close to each other, if not in contact. When having the required practice and experience, one will overcome such difficulties. Learners without such experience tend to break down the long combinations by inserting a short vowel somewhere within the cluster to declusterize it. This declusterization splits the syllable into two syllables that ultimately makes the word easy to pronounce. Declusterization can be attributed to mother tongue negative influence, interference.

The interference of Arabic grammatical structure in English writing is quite normal, as grammatical structure of Arabic, a Semitic language, is very different from that of other Indo-European languages such as English. Arabic language is based on three-consonant root system. All verb forms, nouns, adjectives, participles, etc. are then formed by putting these three-root consonants into fixed vowel patterns, modified sometimes by simple prefixes and suffixes. Not only the irregular patterns of nouns, verbs and adjectives in English confuse the Arabic speakers also the word order ,questions ,negatives ,auxiliaries, verb phrases, verb to-be , pronouns ,articles, adverbs, prepositions and particles, active and passive voice and ,vocabulary .

b) Stage of Development

Language acquisition does not take place at one time but through stages. The learner constructs a system of abstract linguistic rules, which underlies comprehension, and production of the target language; this system is equivalent neither to L2 nor to L1 and referred to as Interlingual communication. At each stage, the learner modifies his/her Interlingual communication by adding rules, deleting rules, or restructuring the whole system. Such modifications are based on the learners' errors; and if the utterance is grammatical, there will be no need for any modification. Certain errors belong to beginning stages while others are found in other stages. Many errors produced by beginners are not found in the Interlingual communication of advanced learners, which means that learners need more time for certain features to master; a fact that reflects their stage of development in their Interlingual communication. One might attribute the pronunciation errors found in (1-20) to the participants stage of development.

Conclusions And Suggestions

As shown in this paper, it is evident that certain English syllables are difficult to learn for Arab learners of English. Although the literature suggested some pronunciation problems which were predictable regarding Arab learners of English in relation to some sounds, the main objective of this paper was to find out the causes that pose pronunciation and grammar difficulties for Arab learners and what makes them declusterize certain English clusters rather than others.

As mentioned above, this paper aims at three main objectives. In relation to objective one, participants did make pronunciation errors in which they declusterize certain target language clusters by inserting an anaptyctic vowel in the onset of some syllables as well as in certain syllable codas.

As far as the second objective is concerned, it is evident from the types of grammatical errors made by the participants that the sources of such difficulties were interference of participants L1 as well as their stage of development. The former was more prominent than the latter.

The third objective was suggesting some teaching procedures that may help teachers as well as learners overcome pronunciation difficulties. The following procedures might be of great assistance when dealing with pronunciation problems related to consonant sequences:

1. Introducing syllable patterns of learners' mother tongue,
2. Introducing short syllable patterns of English language first,
3. Introducing long syllable patterns of English language,
4. Making a comparison between the syllable patterns of both languages pinpointing the differences, and
5. Putting more emphasis on the foreign syllable patterns in order to eliminate the number of pre-edited errors.

References

Barros, A. M. (2003). *Pronunciation difficulties in the consonant system experienced by Arabic speakers when learning English after the age of puberty*, (Unpublished Master Dissertation), West Virginia University, Morgantown: USA.

Al-Shuaibi, A. (2009). *Phonological Analysis of English Phonotactics of Syllable Initial and Final Consonant Clusters by Yemeni Speakers of English*. M.A. Dissertation. Language in India, 9 (11), 195-328.

Altaha, F. (1995). Pronunciation errors made by Saudi university students learning English: Analysis and remedy. *International Review of Applied Linguistics, 109*, 110-123.

Avery, P. & Ehrlich, S. (1992).*Teaching American English pronunciation.* Oxford, England: Oxford University Press

Binturki, T. A. (2008). *Analysis of pronunciation errors of Saudi ESL learners*, (Unpublished Master Dissertation), Southern Illinois University at Carbondale, Carbondale, Illinois: USA.

Bongaerts, T., Planken, B., & Schils, E. (1997). Age and ultimate attainment in the prouniciation of a foreign language. *SLR, 19*, 447-465.

Chiswick, B. & Miller, P. (2007). The critical period hypothesis for language learning: What the 2000 US census says. IZA Discussion Paper No. 2575. Available at SSRN: http://ssrn.com/abstract=961386

Cook, V. J. (1992). Evidence for multi-competence. *Language Learning*, *42(4)*, 557-591.

Eckman, F. (1977). Markedness and the contrastive analysis hypothesis. *Language Learning, 27,* 315-330.

Johnson, J. & Newport, E. (1989). Critical period effects in second language learning: The influence of maturational state on the acquisition of English as a second language. *Cognitive Psychology 21(1)*, 60-99.

Kharma, N. & Hajjaj, A.(1989). *Errors in English among Arabic speakers: Analysis and remedy*, London: Longman.

Lenneberg, E. (1967). *Biological foundations of language*. New York: John Wiley & Sons.

Miller, S. (2000). Looking at progress in a pronunciation class. *TESOL Matters*, *10(2).*

Morley, J. (1991). The pronunciation component in teaching English to speakers of other languages. *TESOL Quarterly*, *25(3),* 481-520.

Oldin, T. (1989). *Language transfer: Cross-linguistic influence in language learning*. Cambridge: Cambridge University Press.

Tushyeh, H. (1996). Linguistic problems facing Arab learners of English. *International Review of Applied Linguistics in Language Teaching*, (1111-1112), (109-117).

Wahba, E. (1998). Teaching pronunciation-why?, *Language Teaching Forum*, *36(3),*3-32.

COMPUTATION OF HYPERGEOMETRIC FUNCTIONS

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Abstract

This research explored some methods for computing the hypergeometric function which can in some cases be difficult to find quickly and accurately. It has found that some softwares, such as *Maple*, are of little use in such instances. So, in this case, this research highlights a method to compute a special case of the hypergeometric function which is $\Phi(a,c,z)=\Phi(\frac{3}{4}-\frac{it}{2},\frac{3}{2},\frac{i\pi\alpha^2 x}{4})$ in a very fast time compared with the *Maple*.

Keywords: hypergeometric function.

Introduction

The calculations of the hypergeometric function ${}_pF_q$ of mathematical physics are often required in many branches of applied mathematics. Despite the importance of this topic, this is sometimes a very hard in practice. The main reason for this is that the function has the complicated structure which produces many interesting mathematical intricacies. The research will focus on computing one of commonly used hypergeometric functions which is ${}_1F_1(a;c;z)=\Phi(a;c;z),$ which is also called Kummer's function as discussed by Abramowitz and Stegun (1972) [1]. Then, It will define the saddle point of a contour integral because it will help to improve the method of solution. In the next section, the research discusses estimation the confluent hypergeometric function using saddle point. The *Maple* software will be used to compute this function. Therefore, the aim of this research is to find a quick and easy method in which we can calculate the values of this function which is a reliable for many different variables. We shall be especially concerned with the case when the magnitude of a and z are large, when special asymptotic formulas have to be developed.

1 Asymptotic expansion of integrals

1.1 The confluent hypergeometric (Kummer's function)

There are many functions defined as special cases of the general confluent hypergeometric function ${}_1F_1(a;b;z)$, including the incomplete gamma function, Modified Bessel functions and Laguerre polynomials as they are suggested by Abramowitz and Stegun (1972) [1]. The research shall be investigating the regular solution (at $z=0$) often denoted by $M(a,b,z)$ (as opposed to $U(a,b,z)$ the irregular solution) of Kummer's differential equation.

$$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}+aw=0. \quad (1)$$

Formally $M(a,b,z)$ is defined by 1 where $M(a,b,z)$ is given by

$$M(a,b,z)=\sum_{s=0}^{\infty}\frac{(a)_s}{(b)_s\, s!}z^s$$

$$=1+\frac{a}{b}z+\frac{a(a+1)}{b(b+1)2!}z^2+\dots \quad (2)$$

where a,b and $z \in \mathbb{C}$ ($b \neq 0,-1,-2,-3,\dots,-n\,;n\in\mathbb{N}$) which is detailed in [1].

The target of this research is not just to use the series 2, but also to find a method to compute this function using *Maple* without using the intrinsic function of Maple ($hypergeom([a],[b],z)$).This special function in *Maple* can calculate the hypergeometric functions in some cases, but sometimes we find that the *Maple* routine is very slow. This is particularly true when $|a|$ or $|z|$ are large compared to $|b|$. However, to begin research shall consider how $M(a,b,z)$ can be calculated directly using the series 2. This series is solved by writing code in *Maple* which calculates the value of the series from $s=0$ to n and termination occurs when *Maple* gets a very small relative error.

1.2 Example

Compute the hypergeometric function using the series 2 if we have:

(i). $a=150I, b=166.0$ and $z=1.1I$

(ii). $a=15000I, b=166.0$ and $z=10000.1I$.

(I means that the number is complex number).

where accurate to a relative error of $\varepsilon = 0.00000001$ by:

(a) Method of Series (Code **A**).

(b) Using Maple's intrinsic function, and then compare the solution in both cases.

First case(i): by the Code **A**, research has got that the value of the confluent hypergeometric function $= [0.369003792014086 - 0.00122244630148484I]$ where $n = 12$ which means that it has taken 12 steps to find the solution with this relative error. By using the intrinsic function in *Mapli. Hypergeom*([150I], [166.0], 1.1I) = [0.369003791936315 − 0.00122244625750552I]. If we compare the two solutions, it is clearly that the relative error quite low which is equal just $-4.4 \times 10^{-11} I$.

In the **second case (ii),** the research had found that the solution can not be found by the series method because the condition $\left[\frac{|a| \times |z|}{|b|}\right] \geq 1$ has not been achieved in the code (Code **A**). While Maple have taken more than two minutes (exactly $132.63s$) to calculate the hypergeometric function. For this reason, research will find another method to compute the hypergeometric function.

2 Saddle points and method steepest descent

Consider the following integral:

$$I(t) = \int_a^b g(x) e^{it\phi(x)} dx. \qquad (3)$$

Here a and $b \in \mathbb{R}$, $a < b$, $t \in \mathbb{R}, t > 1$, and $g(x), \phi(x)$ are real valued ($g(x)$ could be $\in \mathbb{C}$) functions with $g(x)$ defined as that $\int_a^b |g(x)| dx < \infty$ (basically the integral $I(t)$ will exist). Also, $g(x)$ does not contain an exponential term. What research need is a way of obtaining a quick and easy estimate for $I(t)$, with an error term which declines quickly as t gets large. This can be done using the Saddle Point Method of steepest descent, as discussed by Bender and Orszag (1999) [2].

2.1 Saddle points

The first step is to replace x by $z \in \mathbb{C}$ and consider $I(t)$ as a complex integral around some suitable complex contour C.

$$I(t) = \int_C g(z) e^{it\phi(z)} dz.$$

Now assume $\phi(z)$ is a well behaved multi-differentiable, analytic function of $\mathbb{C}$. Applying Taylor's Theorem about a point $z = z_0$ gives :

$$\phi(z) = \phi(z_0) + (z - z_0)\phi'(z_0) + \frac{(z - z_0)^2}{2}\phi''(z_0) + O(z - z_0)^3. \qquad (4)$$

A saddle point of a complex function $\phi(z)$ is a point $z = z_0$ where $\phi'(z_0) = 0$. Suppose $z = z_0$ is saddle point. Then near $z = z_0$ use integral behaves like:

$$I(t) \approx \int_{near z_0} g(z) e^{it[\phi(z_0) + \frac{(z-z_0)^2}{2}\phi''(z_0)]} dz. \qquad (5)$$

The problem with our original integral:

$$I(t) = \int_a^b g(x) e^{it\phi(x)} dx$$

$$= \int_a^b \left(\cos[t\phi(x)] + i\sin[t\phi(x)]\right) g(x) dx, \qquad (6)$$

is that as $t \to \infty$ the $e^{it\phi(x)}$ oscillates faster and faster but does not get any smaller. This means that we have to consider integrating over the whole interual to $x \in [a,b]$ to estimate $I(t)$. If $[a,b]$ is large, such as if $b \to \infty$ this will be a lengthy process, even for a computer. In the case $[a,\infty)$, we also have to worry about how $g(x) \to 0$ as $b \to \infty$ to ensure the integral converges. This may happen quite slowly.

The advantage of the saddle point (steepest descent method) is that it localizes the behaviour of the integral around $z = z_0$ (Bender and Orszag 1999)[2]. This makes the integral much easier to estimate. To do that we will consider:

$$I(t) \approx e^{it\phi(z_0)} \int_{near\, z_0} g(z) e^{\frac{it(z-z_0)^2}{2}\phi''(z_0)} dz, \qquad (7)$$

and assume $\phi''(z_0) > 0$ (the case $\phi''(z_0) < 0$ is easy to deal with change $\frac{\pi}{4}$ to $\frac{-\pi}{4}$ in what follows). To make things easier, assume $z_0 \in \mathbb{R}$ and let C_1 = path in the complex plane (see figure 2.1) such that: $C_1(z) \Rightarrow z = z_0 + \delta e^{i\frac{\pi}{4}}, \delta \in (-R, R)$.

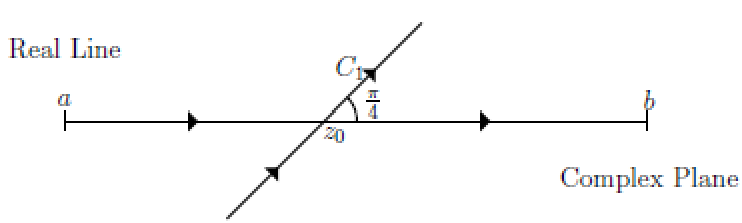

Figure 2.1: The graph of real line and complex plane.

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Clearly this path pases through $z = z_0$ when $\delta = 0$. Along this path we have:

$$g(z)e^{\frac{it(z-z_0)^2}{2}\phi''(z_0)} = g(z_0 + \delta e^{i\frac{\pi}{4}})e^{it(i\delta^2)\frac{\phi''(z_0)}{2}}$$

$$= g(z_0 + \delta e^{i\frac{\pi}{4}})e^{-t\delta^2\frac{\phi''(z_0)}{2}}. \qquad (8)$$

Therefore, the integral for C_1 will be:

$$I(t) = \int_{C_1} g(z)e^{it\phi(z)}dz$$

$$= e^{i\frac{\pi}{4}+it\phi(z_0)}\int_{-R}^{R} g(z_0 + \delta e^{i\frac{\pi}{4}})e^{-t\delta^2\frac{\phi''(z_0)}{2}}d\delta \qquad (9)$$

The function $e^{-t\delta^2\frac{\phi''(z_0)}{2}}$ dies off very rapidly as $|\delta|$ gets large. This means that the $\int_{C_1} dz$ is localised to the area very close to $z = z_0$. The path $z = z_0 + \delta e^{i\frac{\pi}{4}}$ is called Path of Steepest Descant, and along this path, the function $e^{\frac{it(z-z_0)^2}{2}\phi''(z_0)}$ dies off most rapidly. Such steepest descent paths are always associated with saddle points. So, the integral $I(t)$ along C_1 is given by

$$I(t) = e^{i[\frac{\pi}{4}+t\phi(z_0)]}\int_{-R}^{R}\sum_{k=0}^{N}\frac{g^{(k)}(z_0)[\delta e^{i\frac{\pi}{4}}]^k}{k!}e^{-t\delta^2\frac{\phi''(z_0)}{2}}d\delta$$

$$\approx e^{i[\frac{\pi}{4}+t\phi(z_0)]}\left[\frac{\sqrt{\pi}}{\frac{t\phi''(z_0)^{\frac{1}{2}}}{\sqrt{2}}}g(z_0) + \frac{1!!\sqrt{\pi}\,g''(z_0)[e^{i\frac{\pi}{4}}]^2}{\frac{2!(t\phi''(z_0))^{\frac{3}{2}}}{\sqrt{2}}} + \frac{3!!\sqrt{\pi}\,g^{(4)}(z_0)[e^{i\frac{\pi}{4}}]^4}{\frac{4!(t\phi''(z_0))^{\frac{5}{2}}}{\sqrt{2}}} + \dots\right]$$

$$\approx \sqrt{\pi}\, e^{i[\frac{\pi}{4}+t\phi(z_0)]}\sum_{k=0}^{N}\left[\frac{(2k-1)!![t\phi''(z_0)]^{2k}g^{(2k)}(z_0)}{\frac{(2k)![t\phi''(z_0)]^{(2k-1)/2}}{\sqrt{2}}}\right] \qquad (10)$$

This is because we have the fact which is:

$$\int_{-\infty}^{\infty} x^{2n+k} e^{-px^2} dx = \begin{cases} \frac{(2n-1)!!}{(2p)^n}\sqrt{\frac{\pi}{p}} & \text{if } \ k=0, \\ 0 & \text{if } \ k=1. \end{cases} \qquad (11)$$

Provided $[t\phi''(z_0)/2]$ is large, the series 10 will consist initially of rapidly decreasing terms the value of N is used to truncate the series when the terms start to increase. Hence, to first order the integral 9 along C_1 will be

$$I(t); e^{i[\frac{\pi}{4}+t\phi(z_0)]} g(z_0)\sqrt{\frac{\pi}{t\phi''(z_0)/2}} \qquad (12)$$

2.2 Contour Integrals

The integral 3 can be solved by using the saddle point, as stated in [2, 3]. This will be done by forming the path of integration as in the figure 2.2.

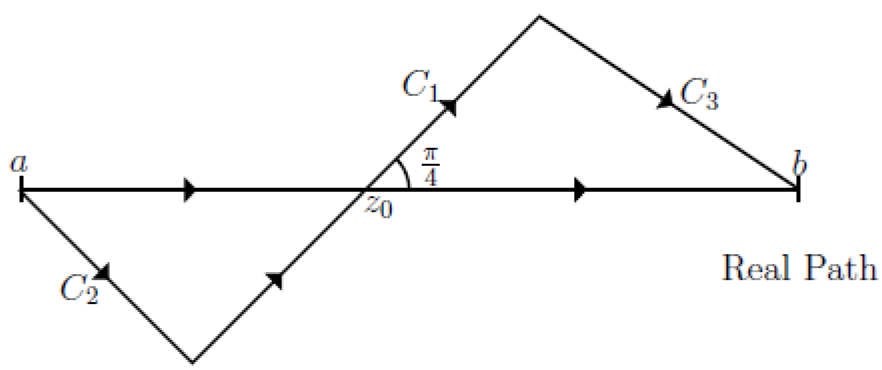

Figure 2.2: Contour of integration in equation 14.

Assuming the function $g(z)$ has no singularities, we can invoke Cauchy's Theorem. Thus:

$$\int_a^b - \int_{C_1} - \int_{C_2} - \int_{C_3} = 0. \qquad (13)$$

Therefore:

$$I(t)=\int_{C_1} g(z)e^{it\phi(z)}dz+\int_{C_2} g(z)e^{it\phi(z)}dz+\int_{C_3} g(z)e^{it\phi(z)}dz \qquad (14)$$

Typically the integrals C_2 and C_3 are smaller than the integral along C_1 because they do not pass through any saddle points of $\phi(z)$. In which case

$$\int_{C2} \text{ and } \int_{C3} \sim \frac{1}{t} \text{ compared to } \int_{C_1} \sim \frac{1}{\sqrt{t}}$$

$$\Rightarrow I(t)=\int_a^b g(x)e^{it\phi(x)}dx \approx e^{i[\pi/4+t\phi(z_0)]}g(z_0)\sqrt{\frac{\pi}{t\phi''(z_0)/2}}+O\left(\frac{1}{t}\right). \qquad (15)$$

2.3 Example

Suppose one has the integral

$$\int_0^{100}\frac{e^{it(x^2-2x)}}{x^2+1}dx \qquad \text{for } t=10,100,1000,\ldots$$

Here $\phi(x)=x^2-2x$, and $g(x)=\dfrac{1}{x^2+1}$ which is well behaved, and dies off sufficiently rapidly for this integral to exist. So this integral will be estimated by using the saddle point method.

$$\phi(x)=x^2-2x \Rightarrow \phi'(x)=2x-2$$

$$\Rightarrow 2x-2=0 \Rightarrow x=1$$

$$\Rightarrow z_0=1 \text{ is the saddle point}$$

$$\Rightarrow \phi''(z_0)=2$$

So near $z_0=1$ using Taylor's Theorem 4 gives $\phi(z)=-1+(z-1)^2$. Set up a steepest descent path $C_1(z)\mapsto z=1+\delta e^{i\frac{\pi}{4}}$ through $z=1$. Then near $z=1$

$$\Rightarrow g(z)=\frac{1}{x^2+1}\sim\frac{1}{2} \text{ so along } C_1$$

$$\Rightarrow \int_{C_1}\frac{e^{it(x^2-2x)}}{x^2+1}dx \approx \frac{1}{2}e^{-it}\int_{-R}^{R}e^{-it\delta^2+\frac{i\pi}{4}}d\delta.$$

Finally, the result for the integral $I(t)$ will be

$$I(t) \approx \frac{\sqrt{\pi}\, e^{-it+\frac{i\sqrt{\pi}}{4}}}{2\sqrt{t}}. \qquad (16)$$

Now the result 16 will be used to compute the hypergeometric function 2 from $t=10$ to 40. The *Maple* has used to compute the result 16 to find the relative error in the approximation from $t=10$ to 40 between the result 16 and the solution by integration directly in *Maple*. In this case, *Maple* has found the integration hard and had taken a long time to compute it when t got large. On the other hand, with the result 16, Maple has computed the integral much faster, as will be shown. If we run the code **B** in *Maple*, we will get the result as in the table 1 which shows some values from that result. Note that T means calculate the time difference between the time for the first solution by the result 16 and the time for the second solution by the integration directory in *Maple*, and ε is the percentage of error between the two solutions.

There are some special cases that have gotten in the table 1. For example, the biggest relative error was when $t=11$. Then, the percent of error decreased step by step until $t=25$ where in this value of t, *Maple* has taken the biggest time to compute the the hypergeometric function ($T=11.123_s$). Suddenly, *Maple* can not compute the hypergeometric function when $t = 26, 27$ and 28 (where the result 14 has computed these values very fast). Therefore, some testes have made for these values to make *Maple* computing them as shown in the code **B**. In another case, the smallest relative error was when $t=29$. After that, *Maple* continues to compute the values until $t=40$.

The figure 2.3 shows us the values of $[t]$ with the time $[T]$ which is clearly that the time for computing the hypergeometric function has increased as t gets large, where the figure 2.4 shows that the relative error has decreased as t gets large.

Table 1: The table of some values of t with times $[T]$ and relative error $[\varepsilon]$.

t	Time taken (T_s)(seconds)	ε
10	2.777	18.96956394%
11 ⋮	4.212 ⋮	20.73015085% ⋮
25	11.123	12.21783312%
26 ⋮	5.585 ⋮	9.981694228% ⋮
29 ⋮	6.209 ⋮	0.1729691650% ⋮
40	6.006	7.007588679%

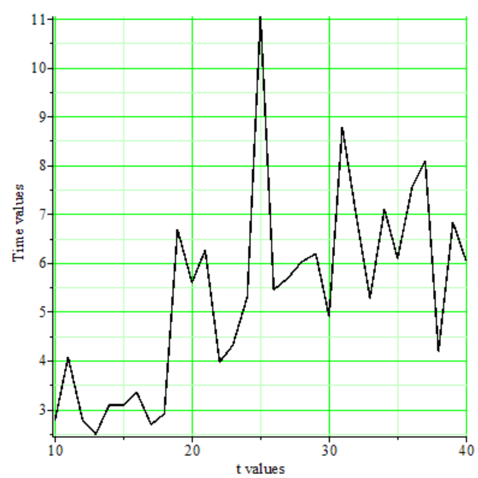

Figure 2.3: Values of $[t]$ With Times $[T]$.

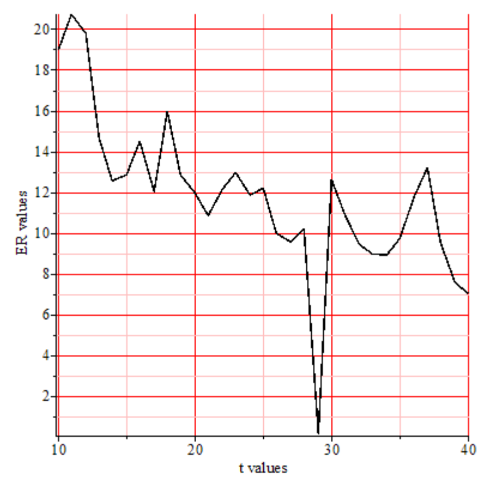

Figure 2.4: Values of $[t]$ with relative error ε.

However, if the code **B** will run with values of t bigger than 40 (for example $t = 60$), *Maple* may not be able to compute the integral. Therefore, section 3 will find another method to compute the hypergeometric function for large values of t.

3 Estimating the Confluent Hypergeometric Function Using Saddle Points

3.1 The analog of Euler's formula

The confluent hyper-geometric function $\Phi(a,c,z)=\Phi(\frac{3}{4}-\frac{it}{2},\frac{3}{2},\frac{i\pi\alpha^2 x}{4})$ plays an important role in the evaluation of the Hardy function $Z(t)$ which gives the amplitude of Riemann's zeta function along the critical line $z=\frac{1}{2}+it$.

Let us look how we might evaluate the more general function $\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4})$. Here both of $t\in\mathbb{R}$ and $\alpha\in\mathbb{R}$ are assumed to be large parameters and also $|t|\gg q, |\alpha|\gg q, q\in\mathbb{R}$ and $q>0$ (for simplicity we can assume $t>0$ and $t<0$ is a simple generalisation). The series representation of

$$\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4})=\sum_{s=0}^{\infty}\frac{(\frac{q}{2}-\frac{it}{2})_n(\frac{i\pi\alpha^2 x}{4})^n}{(q)_s s!} \qquad (17)$$

is totally useless. It would take millions of terms to even make the number start to get smaller if both t and α are large. Consider some other method which will be Euler's integral formula for the confluent hyper-geometric function and it is given by:

$$\Phi(a,c,z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{zx}x^{(a-1)}(1-x)^{(c-a-1)}dx$$

where $Re(c)>Re(a)>0$ and $Re(c)=q$ and $Re(a)=\frac{q}{2}$ which is detailed in [2, 3]. Hence

$$\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4})$$

$$=\frac{\Gamma(q)}{\Gamma(\frac{q}{2}-\frac{it}{2})\Gamma(\frac{q}{2}+\frac{it}{2})}\int_0^1 e^{\frac{i\pi\alpha^2 x}{4}}x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx \qquad (18)$$

The respective Gamma functions are well understood and can be calculated quite easily in *Maple*, using the famous Sterling's series for $\Gamma(z)$, which is valid for large $|z|$. So the key to find an estimate for $\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4})$ is the integral:

$$\int_0^1 e^{\frac{i\pi\alpha^2 x}{4}} x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx \qquad (19)$$

Firstly, let us split up the range of the integral into two parts, $x\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$. Then for the second integral make the substitution,

$$y=(1-x)\Rightarrow x=1-y\Rightarrow dx=-dy \qquad (20)$$

$$\Rightarrow \int_{\frac{1}{2}}^1 e^{\frac{i\pi\alpha^2 x}{4}} x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx$$

$$=\int_{\frac{1}{2}}^0 e^{\frac{i\pi\alpha^2(1-y)}{4}}(1-y)^{(\frac{q}{2}-\frac{it}{2}-1)}y^{(\frac{q}{2}+\frac{it}{2}-1)}(-dy)$$

$$=e^{\frac{i\pi\alpha^2}{4}}\int_0^{\frac{1}{2}} e^{\frac{i\pi\alpha^2(-y)}{4}}(1-y)^{(\frac{q}{2}-\frac{it}{2}-1)}y^{(\frac{q}{2}+\frac{it}{2}-1)}dy. \qquad (21)$$

This means that the integral 19 equals same integral 20 for $x\in(0,\frac{1}{2})$ plus $e^{(\frac{i\pi\alpha^2}{4})}$ multiplied by complex conjugate of same integral for $x\in(0,\frac{1}{2})$. So we just need to know $\int_0^{\frac{1}{2}} e^{\frac{i\pi\alpha^2 x}{4}} x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx$ (since we can easily get its complex conjugate).

Now make a further substitution $x=\frac{1}{w+1}$. This gives us

$$dx=\frac{-1}{(w+1)^2}dw; \qquad w\in(1,\infty)$$

and $\qquad 1-x=\frac{w+1-1}{w+1}\Rightarrow 1-x=\frac{w}{w+1}.$

So the integral becomes

$$\int_\infty^1 e^{\frac{i\pi\alpha^2}{4(w+1)}}\left(\frac{1}{w+1}\right)^{(\frac{q}{2}-\frac{it}{2}-1)}\left(\frac{w}{w+1}\right)^{(\frac{q}{2}+\frac{it}{2}-1)}\frac{-dw}{(w+1)^2}$$

$$= \int_1^{\infty} \frac{e^{\frac{i\pi\alpha^2}{4(w+1)}} \exp[\frac{it}{2}\log(w+1)]\exp[\frac{it}{2}\log(\frac{w}{w+1})]}{w^{1-\frac{q}{2}}(w+1)^q} dw$$

$$= \int_1^{\infty} \frac{\exp[\frac{i\pi\alpha^2}{4(w+1)} + \frac{it}{2}\log(w)]}{w^{1-\frac{q}{2}}(w+1)^q} dw. \qquad (22)$$

Now the denominator behaves like $w^{q+1-\frac{q}{2}} = w^{1+\frac{q}{2}}$ as $w \to \infty$.

Since the numerator is simply a combination of cosines and sines with modulus equal to $1,$ the integral converges for all $q > 0.$ Now how can we evaluate it? This is where we can use the power of the saddle point method. The phase of the numerator

$$if(w) = i\left[\frac{\pi\alpha^2}{4(w+1)} + \frac{t}{2}\log(w)\right]$$

has two large parameters. If we can find the saddle point, we should be able to arrange for the integration path to pass through the saddle point in such a way and the integral can be estimated using an asymptotic series with terms

$$O\left[\left(\frac{1}{Max(\alpha,t)}\right)^{\frac{n}{2}}\right] \qquad n = 1,2,3,\ldots$$

This will converge very rapidly, since α and t are large. Let us consider $f(w)$.

$$f'(w) = \frac{-\pi\alpha^2}{4(w+1)^2} + \frac{t}{2w} = 0$$

$$\Rightarrow \frac{t(w+1)^2}{2} = \frac{\pi\alpha^2 w}{4} \qquad (23)$$

$$\Rightarrow w^2 + 2w + 1 = \frac{\pi\alpha^2 w}{2}$$

$$\Rightarrow w^2 + (2 - \frac{\pi\alpha^2}{2t})w + 1 = 0,$$

$$\Rightarrow w = \frac{-(2-\frac{\pi\alpha^2}{2t}) \pm \sqrt{4+\frac{\pi^2\alpha^4}{4t^2}-\frac{2\pi\alpha^2}{t}-4}}{2}$$

$$= \frac{\pi\alpha^2}{4t} - 1 \pm \frac{\pi\alpha^2}{4t}\sqrt{1-\frac{8t}{\pi\alpha^2}}$$

$$= \frac{2\alpha^2}{a^2} - 1 \pm \frac{2\alpha^2}{a^2}\sqrt{1-\frac{a^2}{\alpha^2}}$$

$$\Rightarrow w = \frac{2\alpha^2}{a^2}\left[1 \pm \sqrt{1-\frac{a^2}{\alpha^2}}\right] - 1$$

where: $\quad a = \sqrt{\frac{8t}{\pi}} \Rightarrow a^2 = \frac{8t}{\pi}.$

Here, the positive square root is only interested. So if $\alpha^2 > a^2$, a real saddle point will be at

$$w_{sad} = \frac{2\alpha^2}{a^2}\left[1 + \sqrt{1-\frac{a^2}{\alpha^2}}\right] - 1 \in [1, \infty).$$

If $\alpha^2 < a^2$, then no real saddle point exists since the square root is complex. So a crucial change in behaviour occurs if $\alpha < a = \sqrt{\frac{8t}{\pi}}$ and $\alpha > a$. Denote w_{sad} by the variable pc or $pc(\alpha)$ and $pc(\alpha > a) \in [1, \infty)$. Differentiating again gives

$$f''(w) = \frac{\pi\alpha^2}{2(w+1)^3} - \frac{t}{2w^2}$$

$$f''(w = pc) = \frac{1}{2}\left[\frac{\pi\alpha^2}{(pc+1)^3} - \frac{t}{pc^2}\right]$$

$$= \frac{t}{2pc^2}\left[\frac{\pi\alpha^2}{t(pc+1)^3} - 1\right]$$

from the equation 23 $\Rightarrow \frac{\pi\alpha^2}{t} = \frac{2(pc+1)^2}{pc}$

$$\therefore f''(w=pc)=\frac{t}{2pc^2}\left[\frac{2pc^2(pc+1)^2}{pc(pc+1)^3}-1\right]$$

$$=\frac{t(pc-1)}{2pc^2(pc+1)}>0,$$

so near the saddle point at $w=pc$

$$f(w)\approx f(pc)+\frac{t(pc-1)}{4pc^2(pc+1)}(w-pc)^2+O((w-pc)^3).$$

So let us consider the integral as contour integral of the form

$$\int_{C_{(z)}}\frac{\exp[i(\frac{\pi\alpha^2}{4(z+1)}+\frac{t}{2}\log(z))]}{z^{1-\frac{q}{2}}(z+1)^q}dz.$$

Suppose we choose as our contour the following in the figure 3.1.

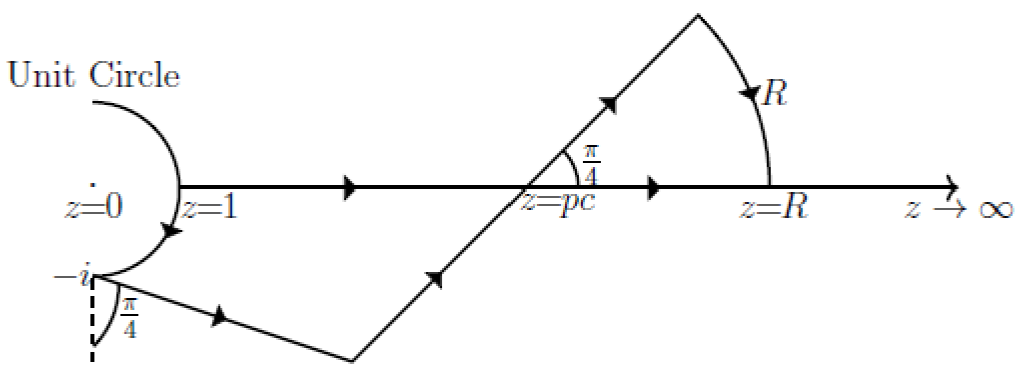

Figure 3.1: Contour of integration in 25.

From the point $z=1,$ move along the unit circle to $z=-i.$ Then draw a line from $z=-i$ at an angle $\frac{\pi}{4}$ $(z=-i+se^{-i\frac{\pi}{4}}, s>0)$ until it crosses the line originating from $z=pc$ at an angle

$\frac{\pi}{4}$ $(z = pc + ue^{i\frac{\pi}{4}}\ u \in (-r, r))$ as shown. Carry on along this ray until it hits the circle centred at $z = 0$ of radius R $(z = Re^{i\theta}\ \theta \geq 0)$. Move along this circle until you reach $z = R$. In the limit $R \to \infty$ this is equivalent to integrating along the real axis from $z = 1$ to $z = \infty$. Since integrand 22 has no poles for $w \geq 1$.

$$\int_1^{\infty} - \int_{C_1(0)} - \int_{(-i+se^{-i\frac{\pi}{4}})} - \int_{(pc+ue^{i\frac{\pi}{4}})} - \int_{C_R(0)} = 0 \qquad (24)$$

by Cauchy's Theorem. Hence the integral 22 is given by

$$\int_1^{\infty} = \int_{C_1(0)} + \int_{(-i+se^{-i\frac{\pi}{4}})} + \int_{(pc+ue^{i\frac{\pi}{4}})} + \int_{C_R(0)} \qquad (25)$$

It would expect that it is the integral through the saddle point that dominates the result. Now consider what the integration $\int_{(pc+ue^{i\frac{\pi}{4}})}$ will equal, where

$$\int_{(pc+ue^{i\frac{\pi}{4}})} = \int_{(pc+ue^{i\frac{\pi}{4}})} \frac{\exp[if(w)]}{w^{(1-\frac{q}{2})}(w+1)^q} dw \qquad (26)$$

Suppose $w = pc + ue^{i\frac{\pi}{4}}; u \in [-r, r] \Rightarrow dw = e^{i\frac{\pi}{4}} du$

i.e near $pc \Rightarrow f(w) = f(pc) + \frac{t(pc-1)}{4pc^2(pc+1)}(w - pc)^2$

$$\Rightarrow f(u) = f(pc) + \frac{iu^2 t(pc-1)}{4pc^2(pc+1)} + O(u^3)$$

So the integral 26 becomes:

$$\approx \int_{-r}^{r} \left[\frac{e^{if(pc)} e^{i\frac{\pi}{4}} \exp[\frac{-u^2 t(pc-1)}{4pc^2(pc+1)}]}{[pc + ue^{i\frac{\pi}{4}}]^{(1-\frac{q}{2})} [pc + 1 + ue^{i\frac{\pi}{4}}]^q} \right] du$$

$$\approx \left(\frac{\exp[\frac{i\pi\alpha^2}{4(pc+1)} + \frac{it}{2}\log(pc) + \frac{i\pi}{4}]}{pc^{(1-\frac{q}{2})}(pc+1)^q} \right) \int_{-r}^{r} e^{-u^2 X(pc)} du$$

where: $X(pc)=\dfrac{t(pc-1)}{4pc^2(pc+1)}$.

Now provided $r^2X(pc)\gg 1$, so that $\exp[-r^2X(pc)]\approx 0$.

We can estimate the integral from tables to give:

$$\left(\frac{2pc\exp[i(\frac{\pi\alpha^2}{4(pc+1)}+\frac{t}{2}\log(pc)+\frac{\pi}{4})]}{pc^{1-\frac{q}{2}}(pc+1)^q}\right)\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]+O\left(\left(\frac{1}{X(pc)}\right)^{\frac{3}{2}}\right)$$

where the value $O\left(\left(\frac{1}{X(pc)}\right)^{\frac{3}{2}}\right)\to 0$ as t gets large and large. Hence the integral 26 becomes:

$$\approx 2\left[\frac{\sqrt{pc}}{pc+1}\right]^q\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]\exp[i(\frac{\pi\alpha^2}{4(pc+1)}+\frac{t}{2}\log(pc)+\frac{\pi}{4})]. \qquad (27)$$

Now if the other integrals in the right hand side from the equation 25 are small, i.e:

$$\left|\int_{C_1(0)}\right|,\left|\int_{C_R(0)}\right| \text{and} \left|\int_{(-i+se^{-i\frac{\pi}{4}})}\right|\ll\left|\int_{(pc+ue^{i\frac{\pi}{4}})}\right|$$

Then our integral 19 equals the result 27 plus $e^{(\frac{i\pi\alpha^2}{4})}$ multiplied by complex conjugate of the result 27 also. Thus,

$$\int_0^1 e^{\frac{i\pi\alpha^2 x}{4}}x^{(\frac{q}{2}-\frac{it}{2}-1)}(1-x)^{(\frac{q}{2}+\frac{it}{2}-1)}dx=$$

$$2\left[\frac{\sqrt{pc}}{pc+1}\right]^q\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]\Bigg[\exp[i(\frac{\pi\alpha^2}{4(pc+1)}+\frac{t}{2}\log(pc)+\frac{\pi}{4})]$$

$$+e^{\frac{i\pi\alpha^2}{4}}\exp[-i(\frac{\pi\alpha^2}{4(pc+1)}+\frac{t}{2}\log(pc)+\frac{\pi}{4})]\Bigg]$$

$$=4e^{\frac{i\pi\alpha^2}{8}}\left[\frac{\sqrt{pc}}{pc+1}\right]^q\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]\cos\left[\frac{\pi}{4}+\frac{t}{2}\log(pc)-\frac{\pi\alpha^2(pc-1)}{8(pc+1)}\right].$$

Hence, research arrives at the final estimate for confluent hypergeometric function 18

$$\Phi(\frac{q}{2}-\frac{it}{2},q,\frac{i\pi\alpha^2 x}{4}) \approx \frac{\Gamma(q)}{\Gamma(\frac{q}{2}-\frac{it}{2})\Gamma(\frac{q}{2}+\frac{it}{2})} 4e^{\frac{i\pi\alpha^2}{8}}\left[\frac{\sqrt{pc}}{pc+1}\right]^q\left[\sqrt{\frac{\pi(pc+1)}{t(pc-1)}}\right]$$

$$\times\cos\left[\frac{\pi}{4}+\frac{t}{2}\log(pc)-\frac{\pi\alpha^2(pc-1)}{8(pc+1)}\right] \qquad (28)$$

where $pc=\frac{2\alpha^2}{a^2}\left[1+\sqrt{1-\frac{a^2}{\alpha^2}}\right]-1\in(1,\infty)$, $a=\sqrt{\frac{8t}{\pi}}$ and $\alpha>a$,

which is valid as $t\to\infty$. Also, if $\alpha\to\infty\Rightarrow pc\approx\frac{4\alpha^2}{a^2}-1\Rightarrow pc+1\approx\frac{4\alpha^2}{a^2}$.

$$\Rightarrow X(pc)=\frac{t(pc-1)}{4pc^2(pc+1)}\approx\frac{t(\frac{4\alpha^2}{a^2}-2)}{4(\frac{4\alpha^2}{a^2}-1)^2(\frac{4\alpha^2}{a^2})}$$

$$\approx\frac{ta^4(4\alpha^2-2a^2)}{16\alpha^2(4\alpha^2-a^2)^2}\approx\frac{ta^4(4-2\frac{a^2}{\alpha^2})}{16\alpha^4(4-\frac{a^2}{\alpha^2})^2}$$

where $\alpha\to\infty\Rightarrow\frac{a^2}{\alpha^2}\to0\Rightarrow X(pc)\approx\frac{t^3}{\pi^2\alpha^4}$, so this should be valid provided $\alpha<t^{\frac{3}{4}}$.

In fact, although we shall not show it here the approximation above is valid for all $\alpha\gg1$ even $\alpha>t^{\frac{3}{4}}$. So we can also take the limit $\alpha\to\infty$.

Now we will consider the integrals which we have ignored in 25. The integrals $\int_{C_R(0)}\to0$ as $R\to\infty$ and $\int_{(-i+se^{-i\frac{\pi}{4}})}$ exponentially quite small. Therefore, we shall show that the integral $\int_{C_1(0)}$ does not contribute to the hypergeometric function. We have the integral

$$\int_{C_1(0)}\frac{\exp[\frac{i\pi\alpha^2}{4(w+1)}+\frac{it}{2}log(w)]}{w^{(1-\frac{q}{2})}(w+1)^q}dw \qquad (29)$$

where $C_1(0)$ is the unit circle. Let us to suppose $w = C_1(0) = e^{i\phi}; \phi \in [0, -\frac{\pi}{2}]$. So

$$\frac{1}{w+1} = \frac{1}{(\cos(\phi)+1) + i\sin(\phi)}$$

$$= \frac{(\cos(\phi)+1) - i\sin(\phi)}{(\cos(\phi)+1)^2 + i\sin^2(\phi)}$$

$$= \frac{\cos(\phi)+1}{2(\cos(\phi)+1)} - \frac{i\sin(\phi)}{2(\cos(\phi)+1)}$$

$$= \frac{1}{2} - \frac{i\sin(\phi)}{2(\cos(\phi)+1)}$$

So the integral 29 becomes

$$ie^{\frac{i\pi\alpha^2}{8}} \int_0^{-\frac{\pi}{2}} \frac{\exp[\frac{\pi\alpha^2 \sin(\phi)}{8(\cos(\phi)+1)} - \frac{t\phi}{2}]e^{i\phi}}{(e^{i\phi})^{(1-\frac{q}{2})}[1+e^{i\phi}]^q} d\phi$$

Note that when ϕ is small and negative, the exponent of the exponential term behaves like $-\frac{1}{2}([\frac{\pi\alpha^2}{8} - t]|\phi|)$ since $\alpha > a$ the exponential term declines very rapidly as ϕ changes from 0 to $-\frac{\pi}{2}$. Now make one further substitution

$$x = \frac{\sin(\phi)}{\cos(\phi)+1} \in [0, -1], \text{ where } \phi \in [0, -\frac{\pi}{2}]$$

$$\Rightarrow \quad \frac{dx}{d\phi} = \frac{\cos(\phi)[1+\cos(\phi)] + \sin(\phi)\times\sin(\phi)}{(1+\cos(\phi))^2}$$

$$= \frac{1}{1+\cos(\phi)} = \frac{1}{2}(1+x^2)$$

$$\Rightarrow \quad d\phi = \frac{2}{(1+x^2)} dx$$

$$\sin(\phi) = x(1+\cos(\phi)) = \frac{2x}{1+x^2} \qquad \Rightarrow 1+\cos(\phi) = \frac{2}{1+x^2}$$

$$\therefore e^{i\phi} = \frac{1}{1+x^2}(1-x^2+2ix)$$

$$\phi = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$1+e^{i\phi} = \frac{1}{1+x^2}(2+2ix)$$

$$= \frac{2}{\sqrt{1+x^2}} e^{i\tan^{-1}(x)}$$

so $$\frac{e^{i\phi}}{(e^{i\phi})^{(1-\frac{q}{2})}(1+e^{i\phi})^q} d\phi = \frac{(e^{i\phi})^{\frac{q}{2}}}{(1+e^{i\phi})^q} d\phi$$

$$= \frac{\exp\left[\frac{iq}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]}{\left(\frac{2}{1+x^2}\right)^q \exp[iq\tan^{-1}(x)]} \times \frac{2}{1+x^2} dx$$

$$= \frac{2^{(1-q)}}{(1+x^2)^{(1-q)}} \times \frac{\exp\left[\frac{iq}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]}{\exp[iq\tan^{-1}(x)]} dx$$

$$= \frac{2^{(1-q)}}{(1+x^2)^{(1-q)}} dx$$

because $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}(x); x \in (-1,1)$. So the integral 29 transforms to:

$$ie^{\frac{i\pi\alpha^2}{8}} \int_0^{-1} \frac{e^{\frac{\pi\alpha^2 x}{8}} \exp[\frac{-t}{2}\arccos\left(\frac{1-x^2}{1+x^2}\right)]}{2^{(q-1)}(1+x^2)^{(1-q)}} dx.$$

Clearly, this integration will equal a real number times $ie^{\frac{i\pi\alpha^2}{8}}$ i.e:

$$ie^{\frac{i\pi\alpha^2}{8}}\int_0^1 \frac{e^{\frac{\pi\alpha^2 x}{8}}\exp[\frac{-t}{2}\arccos\left(\frac{1-x^2}{1+x^2}\right)]}{2^{(q-1)}(1+x^2)^{(1-q)}}dx = ie^{\frac{i\pi\alpha^2}{8}} \times \text{Real Number} \qquad (30)$$

Now in the confluent hypergeometric function, we have found the integral 19 equals same integral for $x \in (0,\frac{1}{2})$ plus $e^{(\frac{i\pi\alpha^2}{4})}$ multiplied by complex conjugate of same integral for $x \in (\frac{1}{2},1)$. However, the contribution around the unit circle $\int_{C_1(0)}$ is given by 29 for the integral $x \in (0,\frac{1}{2})$. So the contribution to the integral 19 from integrals around the unit circle is:

$$ie^{\frac{i\pi\alpha^2}{8}} \times Real\ Number - ie^{\frac{-i\pi\alpha^2}{8}} e^{\frac{i\pi\alpha^2}{4}} \times \text{Same Real Number}$$

$$= ie^{\frac{i\pi\alpha^2}{8}}[\text{Real Number} - \text{Same Real Number}] = 0.$$

So the integral $\int_{C_1(0)}$ in 30 does not contribute the confluent hypergeometric function. Therefore, our confluent hypergeometric function can be estimated very accurately from the saddle point integral through pc along as we will see in example 3.1.1.

3.1.1 Example 1

Compute the hypergeometric function 17 from $t = 500$ to $t = 1000$ where $\alpha = 100$ and $q = 1.5$. *Maple* will be used to find the first solution by the result 28 and for the second solution by Maple's intrinsic function. In the first case, it has used both of the result 28 and *Maple's intrinsic function* to get the time difference between the time for the first solution and the time for the second solution (T). In the second case, the same methods have used but to get the percentage of error (ε). Two graphs are sketched by using these values with the values of t. So, if we run the code **C** in *Maple*, we will get the result as the two graphs below (figures 3.2 and 3.3). The figure 3.2 illustrates dramatically incremental relationship between t and the time T. When t gets large and large, the time difference between the two solutions will increase gradually, which will be more clearly in the next example 3.1.2. While the figure 3.3 shows the inverse relationship between t and the relative error ε. which means if t becomes larger and larger, relative error will fall dramatically. This gives us a very similar result for the computerized result by Maple's intrinsic function but in a very short time.

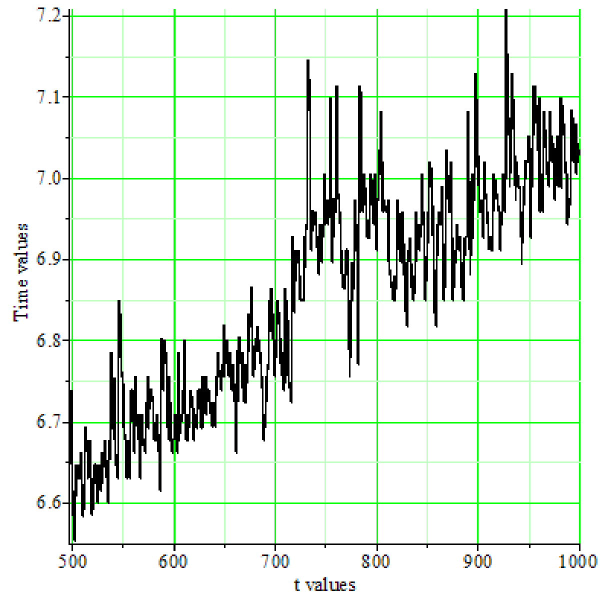

Figure 3.2: Values of $[t]$ With Times $[T]$.

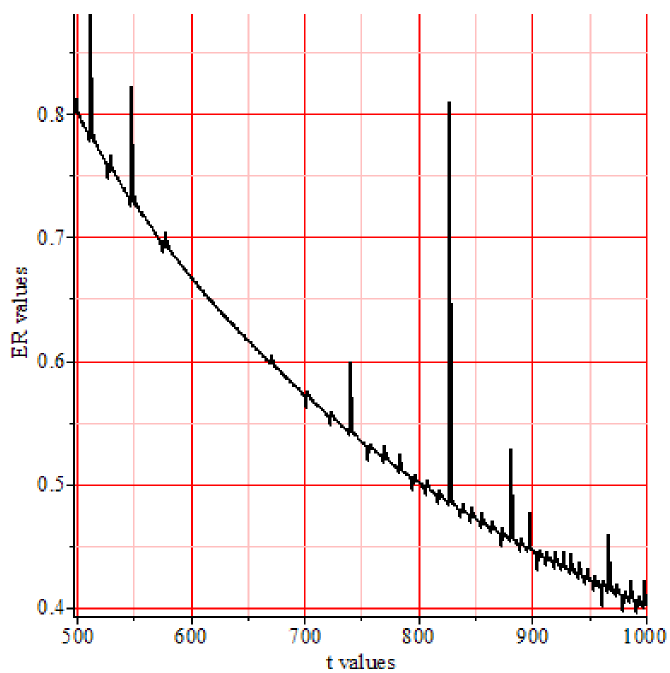

Figure 3.3: Values of $[t]$ with relative error ε.

3.1.2 Example 2

Compute the hypergeometric function 17 if $t = 10000, \alpha = 1000$ and $q = 1.5.$ by:

(a) The result 28. (b) Using Maple's intrinsic function.

In the first case(a), if we run the code **D** in *Maple*, we will get the value of the confluent hypergeometric function $= [1.272100242 \times 10^{6816} + 1.288330276 \times 10^{6812} I]$ and $[T = 0.234_s]$ which means it has taken fractions of a second to calculate the confluent hypergeometric function. However, *Maple* had spent for more than 4 hours to get the result using the case (b) and it did not stop.

4 Conclusion

This research has given a brief overview of the hypergeometric function and its importance in the present time. It focused to compute one of commonly used hypergeometric functions in applied mathematics. So, it has found a quick and easy method to compute this function compared with the *Maple* where this method can help to save time for anyone who is interested in these functions. Thus, future research should be to find the most effective methods to simplify the computation of the hypergeometric function, as it is an important factor in the expansion of applied mathematics, which it will open several other domains in applied science.

References

[1] Abramowitz, M. and Stegun, M. "Handbook of Mathematical function." *Dover Publications.* INC New York. (1972), chapter 13.

[2] Carl M. Bender and Steven A. Orszag. "Advanced Mathematical Method for Scientists and Engineers." *Asymptotic Methos and Perturbation Theory.* INC New York: Spring.(1999), chapter 6.

[3] Yudell L.Luke. "The Special functions and their aproximtions." *Academic Press.* New York and London. (1969), chapter 4.

The code of Maple

There are four codes have written for this research. These codes can be received by sending an email to **heshamalzowam@yahoo.com** .

The list of codes:

1. Code **A** computes the series 2 by using the series method.
2. Code **B** computes the hypergeometric function by using Taylor series 4.
3. Code **C** computes the hypergeometric function by using the result 28.
4. Code **D** is a very similar to the code **C** but it uses when t as a single value.

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Fekete-Szegö inequality for Certain Subclass of Analytic Functions

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Abstract:

In this present work, the author obtain Fekete-Szegö inequality for certain classes of parabolic starlike and uniformly convex functions involving certain generalized derivative operator defined in [1].

1 Introduction

Let A denote the class of all analytic functions in the open unit disk

$$\mathrm{U}=\{z\in \mathbb{C}:|z|<1\},$$

and Let H be the class of functions f in A given by the normalized power series

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k,\quad (z\in \mathrm{U}). \tag{1}$$

Let S denote the class of functions which are univalent in U.

A function f in H is said to be uniformly convex in U if f is a univalent convex function along with the property that, for every circular arc γ contained in U, with centre γ also in U, the image curve $f(\gamma)$ is a convex arc.Therefore, the class of uniformly convex functions is denoted by UCV (see [3]).

It is a common fact from [12], [13] that, for $z\in\mathrm{U}$,that

$$f\in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| < \Re\{1+\frac{zf''(z)}{f(z)}\} \qquad ,(z\in \mathrm{U}). \tag{2}$$

Condition (2) implies that

$$1+\frac{zf''(z)}{f(z)},$$

lies in the interior of the parabolic region

$$R:=\{w: w=u+iv \quad and \quad v^2<2u-1\},$$

for every value of $z \in U$. Let

$$P := \{p : p \in A; p(0) = 1 \quad and \quad \Re(p(z)) > 0; \quad z \in U\},$$

and

$$PAR := \{p : p \in P \quad and \quad p(U) \subset R\}.$$

A function f in H is said to be in the class of parabolic starlike functions, denoted by SP (cf. [13]), if

$$\frac{zf''(z)}{f'(z)} \in R \quad ,(z \in U).$$

Let the functions f given by (1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \ (z \in U),$$

then the Hadamard product (convolution) of f and g, defined by :

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \ (z \in U).$$

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & for \quad k = 0, \\ x(x+1)(x+2)...(x+k-1) & for \quad k \in N = \{1,2,3,...\}. \end{cases}$$

The authors in [1] have recently introduced a new generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n) f(z)$ as the following:

to derive our generalized derivative operator, we define the analytic function

$$\varphi^m(\lambda_1, \lambda_2, l)(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} z^k, \qquad (3)$$

where $m \in N_0 = \{0,1,2,...\}$ and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0$.

Definition 1 *For $f \in A$ the operator $I^m(\lambda_1, \lambda_2, l, n)$ is defined by* $I^m(\lambda_1, \lambda_2, l, n) : A \to A$

$$I^m(\lambda_1, \lambda_2, l, n) f(z) = \varphi^m(\lambda_1, \lambda_2, l)(z) * R^n f(z), \qquad (z \in U), \qquad (4)$$

where $m \in \mathrm{N}_0 = \{0,1,2,...\}$ and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0,$ and $R^n f(z)$ denotes the Ruscheweyh derivative operator [4], and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k, (n \in \mathrm{N}_0, z \in \mathrm{U}),$$

where $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}.$

If $f \in H$, then the generalized derivative operator is defined by

$$I^m(\lambda_1, \lambda_2, l, n) f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k) a_k z^k,$$

where $n, m \in \mathrm{N}_0 = \{0,1,2,...\}$, and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0, c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}.$

Special cases of this operator includes:

- the Ruscheweyh derivative operator [4] in the cases:

$$I^1(\lambda_1, 0, l, n) \equiv I^1(\lambda_1, 0, 0, n) \equiv I^1(0, 0, l, n) \equiv I^0(0, \lambda_2, 0, n)$$

$$\equiv I^0(0,0,0,n) \equiv I^{m+1}(0,0,l,n) \equiv I^{m+1}(0,0,0,n) \equiv R^n,$$

- the S$\hat{a}$l$\hat{a}$gean derivative operator [5]: $I^{m+1}(1,0,0,0) \equiv S^n,$
- the generalized Ruscheweyh derivative operator [6]: $I^2(\lambda_1, 0, 0, n) \equiv R_\lambda^n,$
- the generalized S$\hat{a}$l$\hat{a}$gean derivative operator introduced by Al-Oboudi [7]: $I^{m+1}(\lambda_1, 0, 0, 0) \equiv S_\beta^n,$
- the generalized Al-Shaqsi and Darus derivative operator[8]: $I^{m+1}(\lambda_1, 0, 0, n) \equiv D_{\lambda,\beta}^n,$
- the Al-Abbadi and Darus generalized derivative operator [9]: $I^m(\lambda_1, \lambda_2, 0, n) \equiv \mu_{\lambda_1,\lambda_2}^{n,m},$

and finally

- the Catas drivative operator [10]: $I^m(\lambda_1, 0, l, n) \equiv I^m(\lambda, \beta, l).$

Using simple computation one obtains the next result.

$$(l+1) I^{m+1}(\lambda_1, \lambda_2, l, n) f(z) = (1+l-\lambda_1)[I^m(\lambda_1, \lambda_2, l, n) * \varphi^1(\lambda_1, \lambda_2, l)(z)] f(z) +$$

$$\lambda_1 z[(I^m(\lambda_1,\lambda_2,l,n)*\varphi^1(\lambda_1,\lambda_2,l)(z)]'. \qquad (5)$$

Where $(z\in \mathrm{U})$ and $\varphi^1(\lambda_1,\lambda_2,l)(z)$ analytic function and from (3) given by

$$\varphi^1(\lambda_1,\lambda_2,l)(z)=z+\sum_{k=2}^{\infty}\frac{1}{(1+\lambda_2(k-1))}z^k.$$

Definition 2 *Let $SP^m(\lambda_1,\lambda_2,l,n)$ be the class of functions $f\in H$ satisfying the inequality :*

$$\left|\frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}-1\right|<\Re\{\frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\},\quad (z\in \mathrm{U}). \qquad (6)$$

Note that many other operators are studied by many different authors, see for example [19, 20, 21]. There are times, functions are associated with linear operators and create new classes (see for example [18]). Many results are considered with numerous properties are solved and obtained.

However, in this work we will give sharp upper bounds for the Fekete-Szegö problem. It is well known that Fekete and Szegö [14] obtained sharp upper bounds for $|a_3-\mu a_2^2|$ for the case $f\in S$ and μ is real. The bounds have been studied by many since the last two decades and the problems are still being popular among the writers. For different subclasses of S, the Fekete-Szegö problem has been investigated by many authors including [14, 12, 15, 16, 17], few to list. For a brief history of the Fekete-Szegö problem see [17].In the present paper we completely solved the Fekete-Szegö problem for the class $SP^m(\lambda_1,\lambda_2,l,n)$ defined by using $I^m(\lambda_1,\lambda_2,l,n)$.

2 Fekete-Szegö problem for the class $SP^m(\lambda_1,\lambda_2,l,n)$

Here we obtain sharp upper bounds for the Fekete-Szegö functional $|a_3-\mu a_2^2|$ for functions f belonging to the class $SP^m(\lambda_1,\lambda_2,l,n)$,

Let the function f, given by

$$f(z)=z+a_2z^2+a_3z^3+...,(z\in \mathrm{U}), \qquad (7)$$

be in the class $SP^m(\lambda_1,\lambda_2,l,n)$. Then by geometric interpretation there exists a function w satisfying the conditions of the Schwarz ' lemma such that

$$\frac{z(I^m(\lambda_1,\lambda_2,l,n)'f(z))}{I^m(\lambda_1,\lambda_2,l,n)f(z)}=q(w(z)),\quad (z\in \mathrm{U}).$$

It can be verified that the Riemann map q of U onto the region R, satisfying $q(0)=1$ and $q_0(0)>0,$ is given by

$$q(z)=1+\frac{2}{\pi^2}(log\frac{1+\sqrt{z}}{1-\sqrt{z}})^2,$$

$$=1+\frac{8}{\pi^2}\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{2k+1})z^n,$$

$$=1+\frac{8}{\pi^2}(z+\frac{2}{3}z^2+\frac{23}{45}z^3+\frac{44}{105}z^4+...), \qquad (z\in \mathrm{U}).$$

Let the function P_1 in P be defined by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots, \qquad (z\in \mathrm{U}).$$

Then by using

$$w(z)=\frac{p_1(z)-1}{p_1(z)+1},$$

we obtain

$$a_2=\frac{4(1+l)^{m-1}(1+\lambda_2)^m}{\pi^2(1+\lambda_1+l)^{m-1}(n+1)}c_1,$$

and

$$a_3=\frac{4(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}(c_2-\frac{c_1^2}{6}(1-\frac{24}{\pi^2})).$$

These expressions shall be used throughout the rest of the paper.

In order to prove our result we have to recall the following lemmas:

Lemma 1 *[11] If* $p_1(z)=1+c_1z+c_2z^2+\cdots$ *is an analytic function with positive real part in* U, *then*

$$|c_2-\upsilon c_1^2|\le\begin{cases}-4\upsilon+2 & if \quad \upsilon\le 0,\\ 2 & if \quad 0\le\upsilon\le 1,\\ 4\upsilon+2 & if \quad \upsilon\ge 1.\end{cases}$$

When $\upsilon < 0$ or $\upsilon > 1$, the equality holds if and only if $p_1(z)$ is $\frac{(1+z)}{(1-z)}$ or one of its rotations. If $0 < \upsilon < 1$, then the equality holds if and only if $p_1(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\upsilon = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}a\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}a\right)\frac{1-z}{1+z} \quad (0 \le a < 1),$$

or one of its rotations. If $\upsilon = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\upsilon = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < \upsilon < 1$:

$$|c_2 - \upsilon c_1^2| + \upsilon |c_1| \le 2, \qquad (0 < \upsilon \le \frac{1}{2}),$$

and

$$|c_2 - \upsilon c_1^2| + (1-\upsilon)|c_1| \le 2, \qquad (\frac{1}{2} < \upsilon \le 1).$$

Lemma 2 *[2] Let* h *be analytic in* U *with* $\Re\{h(z)\} > 0$ *and be given by* $h(z) = 1 + c_1 z + c_2 z^2 + \ldots,$ *for* $z \in \mathrm{U}$*, then*

$$|c_2 - \frac{c_1^2}{2}| \le 2 - \frac{|c_1|^2}{2}.$$

Lemma 3 *[11] Let* $h \in P$ *where* $h(z) = 1 + c_1 z + c_2 z^2 + \cdots.$

Then $$|c_n| \le 2, \quad n \in \mathrm{N},$$

and $$|c_2 - \frac{1}{2}\mu c_1^2| \le 2 + \frac{1}{2}(|\mu - 1| - 1)|c_1|^2.$$

Theorem 1 *If* f *be given by (1) and belongs to the class* $SP^m(\lambda_1, \lambda_2, l, n)$. *Then,* $|a_3 - \mu a_2^2| \le$

$$\begin{cases} \frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}[\frac{4\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1(k-1)+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{1}{3}-\frac{4}{\pi^2}] & if \quad \mu\leq\sigma_1, \\ \frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)} & if \quad \sigma_1\leq\mu\geq\sigma_2, \\ \frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}[\frac{1}{3}+\frac{4}{\pi^2}-\frac{4\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1(k-1)+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}] & if \quad \mu\leq\sigma_2, \end{cases} \tag{8}$$

where

$$\sigma_1=\frac{(1+2\lambda_2)^m(1+\lambda_1+l)^{2(m-1)}(n+1)}{(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)^{m-1}(n+2)}(1+\frac{5\pi^2}{24}), \tag{9}$$

$$\sigma_2=\frac{(1+2\lambda_2)^m(1+\lambda_1+l)^{2(m-1)}(n+1)}{(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)^{m-1}(n+2)}(1-\frac{\pi^2}{24}). \tag{10}$$

each of the estimates in (8) is sharp.

Proof: An easy computation shows that

$$|a_3-\mu a_2^2|=\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left|\left(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{1}{3}-\frac{8}{\pi^2}\right)c_1^2-2c_2\right| \tag{11}$$

$$\leq\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left[\left(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{5}{3}-\frac{8}{\pi^2}\right)|c_1|^2+2|c_1^2-c_2|\right]. \tag{12}$$

If $\mu\geq\sigma_1$, then the expression inside the first modulus on the right-hand side of (12) is nonnegative.

Thus,by applying Lemma 3 ,we get

$$=\frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left[\left(\frac{4\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{1}{3}-\frac{4}{\pi^2}\right)\right], \tag{13}$$

which is the assertion (8). Equality in (13) holds true if and only if $|c_1|=2.$ Thus the function f is $k(z;0;1)$ or one of its rotations for $\mu>\sigma_1.$

Next, if $\mu\le\sigma_2,$ then we rewrite (11) as

$$|a_3-\mu a_2^2|=\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left|\left(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}+\frac{1}{3}-\frac{8}{\pi^2}\right)c_1^2-2c_2\right|$$

$$\le\frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left|\left(\frac{1}{3}+\frac{8}{\pi^2}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right)\right|.$$

The estimates $|c_2|\le 2$ and $|c_1|\le 2,$ after simplification, yield the second part of the assertion (8), in which equality holds true if and only if f is a rotation of $k(z;0;1)$ for $\mu<\sigma_2,.$ If $\mu=\sigma_2,,$ then equality holds true if and only if $|c_2|=2.$ In this case, we have

$$p_1(z)=\left(\frac{1+\nu}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\nu}{2}\right)\frac{1-z}{1+z}\quad(0\le\nu<1;z\in\mathrm{U}).$$

Therefore the extremal function f is $k(z;0;\nu)$ or one of its rotations.

Similarly, $\mu=\sigma_1,$ is equivalent to

$$\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{5}{3}-\frac{8}{\pi^2}=0.$$

Therefore, equality holds true if and only if $|c_1^2-c_2|=2.$

This happens if and only if

$$\frac{1}{p_1(z)}=\left(\frac{1+\nu}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\nu}{2}\right)\frac{1-z}{1+z}\quad,(0\le\nu<1;z\in\mathrm{U}).$$

Thus the extremal function f is $k(z;\pi;\nu)$ or one of its rotations.

Finally, we see

$$|a_3-\mu a_2^2|=\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left|2(c_2-\frac{1}{2}c_1^2)+\left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right)c_1^2\right|,$$

and

$$\max\left|\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right|\le 1, \qquad (\sigma_1\le\mu\ge\sigma_2).$$

Therefore, using Lemma 3, we get

$$|a_3-\mu a_2^2|\le\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[2(2-\frac{1}{2}|c_1|^2)+|c_1|^2\right]$$

$$=\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)} \qquad ,if \qquad \sigma_1\le\mu\ge\sigma_2.$$

If $\sigma_1<\mu<\sigma_2$, then equality holds true if and only if $|c_1|=0$ and $|c_2|=0$. Equivalently, we have

$$p_1(z)=\frac{1+\nu z^2}{1-\nu z^2} \quad ,(0\le\nu\le 1; z\in \mathrm{U}).$$

Thus the extremal function f is $k(z;0;0)$ or one of its rotations.

3 IMPROVEMENT OF THE ESTIMATION

Theorem 2 *If $\sigma_1\le\mu\le\sigma_2$, then in view of Lemma , Theorem can be improved as follows:*

$$|a_3-\mu a_2^2|+\left(\mu-\frac{(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}{(1+l)^{m-1}(1+\lambda_2)^m(1+2\lambda_1+l)^{m-1}(n+2)}(1-\frac{\pi^2}{24})\right)|a_2|^2$$

$$\le\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1(k-1)+l)^{m-1}(n+1)(n+2)} \quad (\sigma_2\le\mu\le\sigma_3),$$

and

$$|a_3-\mu a_2^2|+\left(\frac{(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}{(1+l)^{m-1}(1+\lambda_2)^m(1+2\lambda_1+l)^{m-1}(n+2)}(1+\frac{5\pi^2}{24})-\mu\right)|a_2|^2$$

$$\leq\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1(k-1)+l)^{m-1}(n+1)(n+2)}\quad(\sigma_3\leq\mu\leq\sigma_1),$$

where σ_1 and σ_2 are given, as before, by (9), (10), and

$$\sigma_3=\frac{(1+2\lambda_2)^m(1+\lambda_1+l)^{2(m-1)}(n+1)}{(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)^{m-1}(n+2)}(1+\frac{\pi^2}{12}).$$

Proof: For the values of $\sigma_1\leq\mu\leq\sigma_3,$ and from Lemma 2 we have

$$|a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2|^2\leq$$

$$\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[2(2-\frac{1}{2}|c_1|^2)+\left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right)|c_1|^2\right]$$

$$+\frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^4(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[\frac{\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(n+2)}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)}-(1-\frac{\pi^2}{24})\right]|c_1|^2$$

$$\leq\frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}[4-|c_1|^2+\left(\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right)|c_1|^2$$
$$+\left(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{8}{\pi^2}+\frac{1}{3}\right)|c_1|^2]$$

$$=\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}.$$

Similarly, if $\sigma_2\leq\mu\leq\sigma_3,$ we can write

$$|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2|^2\leq \frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}$$

$$\left[2(2-\frac{1}{2}|c_1|^2)+(\frac{8}{\pi^2}+\frac{2}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2(k-1))^m(n+2)}{\pi^2(1+\lambda_1(k-1)+l)^{m-1}(n+1)})|c_1|^2\right]$$

$$+\frac{16(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^4(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}\left[1+\frac{5\pi^2}{24}-\frac{\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}\right]|c_1|^2$$

$$\leq \frac{2(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}[4-|c_1|^2+(\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)}-\frac{8}{\pi^2}-\frac{2}{3})|c_1|^2$$

$$+(\frac{8}{\pi^2}+\frac{5}{3}-\frac{8\mu(1+l)^{m-1}(1+\lambda_2)^{2m}(1+2\lambda_1+l)(n+2)}{\pi^2(1+\lambda_1+l)^{2(m-1)}(1+2\lambda_2)^m(n+1)})|c_1|^2]$$

$$=\frac{8(1+l)^{m-1}(1+2\lambda_2)^m}{\pi^2(1+2\lambda_1+l)^{m-1}(n+1)(n+2)}.$$

References

[1] A. A. Amer and M. Darus, On some properties for new generalized derivative operator, *Jordan Journal of Mathematics and Statistics (JJMS)*, **4**(2) (2011), 91-101.

[2] Ch. Pommerenke, Univalent functions, Vandenhoeck and Ruprecht, G¨ottingen, (1975).

[3] A.W. Goodman, On uniformly convex functions, Ann. Polon. Math. 56, 87-92 (1991).

[4] St. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* Vol. 49, 1975, pp. 109-115.

[5] G. S.Sâlâgean, Subclasses of univalent functions, *Lecture Notes in Math*. (Springer-Verlag), 1013, (1983), 362-372.

[6] K. Al-Shaqsi and M. Darus, An operator defined by convolution involving polylogarithms functions, *J. Math. Stat.*, 4(1), (2008), 46-50.

[7] F.M. AL-Oboudi, On univalent functions defined by a generalised Sâlâgean Operator, *Int, J. Math. Math. Sci*. 27, (2004), 1429-1436.

[8] K. Al-Shaqsi and M. Darus, Differential Subordination with generalised derivative operator, *Int. J. Comp. Math. Sci*, 2(2)(2008), 75-78.

[9] M. H. Al-Abbadi and M. Darus,Differential Subordination for new generalised derivative operator, *Acta Universitatis Apulensis*, 20, (2009),265-280 .

[10] A. Catas ,On a Certain Differential Sandwich Theorem Associated with a New Generalized Derivative Operator, *General Mathematics.* 4 (2009), 83-95.

[11] W. Ma and D. Minda, "A unifed treatment of some special classes of univalent functions ,in: Proceedings of the Conference on Complex Analysis, Z. Li, F. Ren,L. Yang, and S. Zhang(Eds.), Internat. Press (1994), 157-169.

[12] W. Ma and D. Minda, Uniformly convex functions, Ann. Polon. Math. 57, 165-175 (1992).

[13] E. Rϕnning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. 118(1993), 189-196 .

[14] M. Fekete, G. Szegö, Eine Bemerkung über ungerade schlichte Funktionen, J. London Math. Soc. 8 (1933) 85-89.

[15] H.M. Srivastava, A.K. Mishra, Applications of fractional calculus to parabolic starlike and uniformly convex functions, Comput. Math. Appl. 39 (3–4)(2000) 57-69.

[16] M.Darus, I. Faisal and M.A.M. Nasr, Differential subordination results for some classes of the family $\zeta(\upsilon,\theta)$ associated with linear operator, Acta Univ. Sapientiae, MATHEMATICA, 2(2) (2010), 184-194.

[17] M.A. Al-Abbadi and M. Darus. Differential subordination defined by new generalised derivative operator for analytic functions, Inter. Jour. Math. Math. Sci. 2010 (2010), Article ID 369078, 15 pages.

[18] S. F. Ramadan and M.Darus. Generalized differential operator defined by analytic functions associated with negative coefficients, Jour. Ouality Measurement and Analysis, 6(1) (2010), 75-84. 59.

[19] R. W. Ibrahim and M. Darus. On univalent function defined by a generalized differential operator. Journal of Applied Analysis (Lodz), 16(2) (2010), 305-313.

[20] Aisha Ahmed Amer , Second Hankel Determinant for New Subclass Defined by a Linear Operator, Springer International Publishing Switzerland 2016, Chapter 6 .

[21] Aisha Ahmed Amer , Properties of Generalized Derivative Operator to A Certain Subclass of Analytic Functions with Negative Coefficients,.2017 ، جامعة بنغازي،كلية التربية، المجلة الليبية العالمية

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Antimicrobial Activities of Six Types of Honey towards Selected Pathogenic Microorganisms

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Abstract

The emergence of strains of pathogenic microorganisms with resistance to commonly used antibiotics has necessitated a search for novel types of antimicrobial agents. It is of a great importance to understand the efficiency of honey against microorganisms. The main objective of this study was to investigate the biological activities of Black Seed (Nigella sativa) honey, Thymus honey, Myrrh honey, Tamarix honey, Spring honey and Zizphus honey before and after exorcism against four pathogenic bacteria and one pathogenic fungi. Agar diffusion, Minimum Inhibitory Concentration and Minimum Killing Concentration methods were used to evaluate the antimicrobial activity. All the Gram-positive and Gram-negative bacteria tested were found to be inhibited to some extent by the six different honeys, although the antimicrobial potency was highly dependent upon type of honey and test organism. The honey samples were especially active against methicillin-resistance Staphylococcus aureus (isolated from Tripoli Medical Center), S. aureus ATCC 43300), Pseudomonas. aeruginosa ATCC 27893, . Escherichia coli ATCC 35150 and the yeast Candida albicans that isolated from Al-jala hospital. S. aureus was found to be particularly sensitive to Thymus exorcism honey. However, C. albicans found to be sensitive to Spring honey. In addition, the results showed that Tamarix exorcism honey was significantly ($P = 0.006$) more active than the Tamarix honey, also N. sativa exorcism honey was significantly ($P = 0.003$) more active than the N. sativa honey. The Minimum Inhibitory Concentration (MIC) of Tamarix exorcism honey (determined by tube dilution) for MRSA is more effective than 32-fold lower at 3.125% (V/V) compared to the other five honeys. The Minimum killing concentration was 6.25% concentration against S. aureus. The honey samples studied proved to be a good source of antimicrobial agents that might serve to fight against several diseases. These data also demonstrate that antimicrobial potency of honeys is highly dependent upon both the target microbial species and the type of honey.

1. Introduction:

Honey is the natural sweet substance produced by honey bees, Apis mellifera, from the nectar of plants (blossoms) or from the secretions of living parts of plants or excretions of plant sucking insects on the living parts of plants, which honey bees collect, transform by combining with specific substances of their own, deposit, dehydrate, store and leave in the honey comb to ripen and mature (Codex Alimentarius Commission (2001a, b)).

It has been possesses inherent antimicrobial properties, some of which are due to high osmotic pressure/low water activity, in which the low water activity of honey is inhibitory to the growth of the majority of bacteria, and to many yeasts and moulds. When applied topically to wounds, osmosis would be expected to draw water from the wound into the honey, helping to dry the infected tissue and reduce bacterial growth. Even when diluted with water absorbed from wounds, honeys would be likely to retain a water activity sufficiently low to inhibit growth of most bacteria. Honey is mildly acidic, with a pH between 3.2 and 4.5. Gluconic acid is formed in honey when bees secrete the enzyme glucose oxidase, which catalyses the oxidation of glucose to gluconic acid, the low pH alone is inhibitory to many pathogenic bacteria and, in topical applications at least, could be sufficient to exert an inhibitory effect (Molan 1995).

Hydrogen peroxide, the end product of the glucose oxidase system and tetracycline derivatives has the antibacterial properties against pathogens. Low concentrations of this known antiseptic are effective against infectious bacteria and can play a role in the wound healing mechanism and in Stimulation and proliferation of peripheral blood lymphocytic and phagocytic activity. Other factors, such as low protein content, high carbon to nitrogen ratio, low redox potential due to the high content of reducing sugars, viscosity/anaerobic environment and other chemical agents/phytochemicals are also likely to play some role in defining antibacterial activity of honey.

Therapy with bee products as honey is an old tradition and honey has had many therapeutic uses from ancient times to the present. It has been suggested that pure honey is bactericidal for many pathogenic organisms, including various gram negative and gram positive bacteria. Other therapeutic effects of honey include its use in the treatment of fungal infections, burns, infantile gastroenteritis wounds and decubitus ulcers (Bergman et al. 1983).

It is known that honey strongly inhibits the growth of microorganisms. Already in 1892, the Dutch scientist Van Ketel demonstrated that honey has bactericidal effects. A great number of research reports have subsequently confirmed his findings. (Molan 1999) found that honey is becoming accepted as a reputable and effective therapeutic agent by practitioners of conventional medicine and by the general public. This is because of good clinical results that are being obtained. Honey has been reported to be effective in the healing of infected postoperative wounds. It has also been reported to inhibit the growth of many bacteria such as Bacillus cereus, Staphylococcus aureus, Salmonella dublin, and Sh. dysenteriae. It has also been reported to inhibit the growth of Bacteroides spp. (Elbugoury and Rasomy, 1993).

The aim of this study was to determine, the antimicrobial of six types of honey which representative (Thymus honey, Tamarix honey, N. sativa honey, Spring honey, Zizphus honey and Myrrh honey) and also to compare the antimicrobial of honey before and after exorcism against four types of bacteria Staphylococcus aureus, Methicillin-resistant Staphylococcus aureus as gram-positive, Escherichia coli, Pseudomonas aeruginosa as gram-negative and one fungi (Candida albicans).

2. Materials and method:

2.1- Honey samples:

The samples of Thymus floral and zizphus floral (originating from Albyda), Tamarix floral, and Spring (originating from Yafran), Myrrh floral (originating from Gebel Akhdar) and N. sativa floral honeys were purchased from local shops in Tripoli (Libya).

Table 1: Examined natural honey samples and their floral sources.

Honey sample	Scientific name of plant cover	Aceae	Season
Thymus	Thymus capitatus (L.)	Limaceae	summer
Zizphus	Zizphus lotus Desf.	Rhamnaceae	Summer
Tamarix	Tamarix aphylla (L.) Karst	Tamaricaceae	Spring
Nigella staiva	Nigella staiva L.	Ranunculaceae	Spring
Myrrh	Arbutus pavarii	Ericacees	Autmen
Spring	Acacia cyanophylla Citrus limon (L.) Burm	Fabaceae Rutaceae	Spring

Muller Hinton Broth and Muller Hinton Agar (Oxoid) were cultivation of bacteria. Sabouraud Dextrose Broth and Sabouraud Dextrose Agar (Oxoid) were used for cultivation of fungi. All these culture media were obtained from Sigma-Aldrich Company.

2.2- Honey preparation and In Vitro Antimicrobial Assay:

Four bacterial species obtained from faculty of Medicinal, University of Tripoli (Libya) were selected for testing. Representative strains of Gram-positive bacteria were S. aureus ATCC 43300 and MRSA (Isolated from Tripoli Medical Center. Representative strains of Gram-negative bacteria were E. coli ATCC 35150 and P. aeruginosa ATCC 27893.

Representative fungal (yeast) strain was C. albicans (Isolated from Al-jala Hospital). The honeys samples were stored in the dark at room temperature. For agar well diffusion tests, the honeys were initially diluted in de-ionized water to obtain concentrations of 10%, 25%, 50 and 75% Wt/Vol.

The plates were prepared using 50ml of sterile Muller Hinton Agar. The surface of the plates was aseptically inoculated using a 100µl of suspension of bacteria grown overnight at 37ºC in Muller Hinton Agar Broth and allowed to dry. Agar cylinders of, 7mm in diameter were cut from the culture media using a sterile glass Derhum tube; and then filled with the dilutions of honeys. The plates were incubated at 37ºC and observed after 24 hours; the clear, circular inhibition zones around the wells were measured. For MIC determination, the honey samples were initially diluted in Muller Hinton Broth 100% (Vol/Vol). Six serial two fold dilution of the preparation were made in Muller Hinton Broth for bacterial inhibition testing; the lowest concentration was applied is 3.125% (Vol/Vol). Control comprised broth only (negative) or test organism and broth (positive control) were prepared.

2.3 Antibiotic Susceptibility Test:

Antibiotic susceptibility on organisms were studied against the Ciprofloxacin 10 µg and ketoconazole 10ug compared of concentrations of honey by the disk diffusion technique on (MHA), using inhibition zone criteria recommended by the disk manufacturer and based on the method of (Barry 1976). The selection of antibiotic disks was performed according to the guidelines recommended by ATCC.

3.-Result:

The six types of honey used in this study were inhibited the growth of all the bacteria and fungi tested. The antimicrobial activities of all types of honey on the different bacteria and fungi strains are tested is shown in Tables 2, 3, 4, 5, 6 and 7. The average zone of inhibition of honey against the strains ranged from 0.0 ± 0.0 (all strains) to 21 ± 0.88 (P. aeruginosa).

Differences regarding inhibition were observed for the Thymus honey before and after exorcism (Table 2). Thymus exorcism honey was more active with the largest inhibition against S. aureus and MRSA organisms. Thymus exorcism honey showed marked inhibition of growth on S. aureus, the maximum inhibition was shown at concentration of 100% as 21 ± 0.66 mm, which reduced to 16 ± 0.57 mm at 75% mm and 15 ± 0.33 mm at 50% concentration. While, Thymus honey showed marked inhibition of growth on S. aureus, the inhibition was shown at concentration of 100% as 17 ± 0.88 mm (Table 2).

Table 2: Antimicrobial activities of the Thymus honey before and after exorcism. **For each row,** Different letters after SE values indicate statistical significant differences between means (P < 0.05).

Organisms	Mean diameter of Inhibition Zone (mm) ± SE							
Honey	Thymus honey				Thymus exorcism honey			
	25%	50%	75%	100%	25%	50%	75%	100%
Gram- Positive								
S. aureus	0 ± 0	0 ± 0	0 ± 0	17 ± 0.88	0 ± 0	15 ± 0.33*	16 ± 0.57*	21 ± 0.66
S. aureus (MRSA)	0 ± 0	0 ± 0	0 ± 0	15 ± 0.88	0 ± 0	12 ± 1.0*	15 ± 0.88*	19 ± 0.57
Gram- Negative								
E. coli	0 ± 0	0 ± 0	0 ± 0	13 ± 0.88	0 ± 0	0 ± 0	0 ± 0	15 ± 1.0
P. aeruginosa	0 ± 0	0 ± 0	0 ± 0	19 ± 0.58*	0 ± 0	0 ± 0	0 ± 0	12 ± 0.88
Fungi								
C. albicans	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	12 ± 0.88*

*Significant of compared to two values in the same row.

Also the table showed that MRSA growth with inhibition zone at concentration of 100% as 19 ± 0.57 mm, which reduced to 15 ± 0.88 mm at 75% and 12 ± 1.0 mm at 50% concentration for Thymus exorcism honey and at concentration of 100% as 15 ± 0.88 mm for Thymus honey. The inhibition of growth on E. coli was shown at concentration of 100% as 15 ± 1.0 mm for Thymus exorcism honey and at the same concentration as 13 ± 0.88 mm for Thymus honey. Inhibition of growth on P. aeruginosa, the inhibition zone was at concentration of 100% as 19 ± 0.58 mm. However, Thymus exorcism honey only was inhibited the growth of C. albicans zone at concentration of 100% as 12 ± 0.88 mm in comparison with Thymus honey. However, no effect was observed at concentrations of 25%, 50%, 75% and 100% for Thymus honey.

Table 3 showed marked of inhibition on P. aeruginosa, the maximum zone was shown at concentration of 100% as 21 ± 0.66 mm, which reduced to 19 ± 0.57 mm at 75%, 18 ± 0.88 mm at 50% and 16 ± 1.20 at 25% concentration for the Tamarix exorcism honey which compared to the Tamarix honey at concentration of 100% at zone of 18 ± 0.58 mm. Also the table exhibited that MRSA grow with inhibition zone at concentration of 100% as 18 ± 1.76 mm, which reduce to 14 ± 0.88 mm at 75% and 11 ± 0.88 mm at 50% concentration for Tamarix exorcism honey and the Tamarix honey was inhibition at only concentration of 100% as 16 ± 0.88 mm. S. aureus shows inhibition zone with Tamarix honey as 19 ± 1.0 mm at 100% and with Tamarix exorcism honey was 17 ± 1.45 mm at the same concentration. E. coli shows inhibition zone with Tamarix exorcism honey as 12 ± 0.88 mm at 100% concentration and with Tamarix honey was 11 ± 0.66 mm. C. albicans showed inhibition zone at concentration of 100% as 13 ± 0.33 mm for Tamarix honey.

Table 3: Antimicrobial activities of the Tamarix honey before and after exorcism. **For each row,** Different letters after SE values indicate statistical significant differences between means ($P < 0.05$).

Organisms	Mean diameter of Inhibition Zone (mm) ± SE							
Honey	Tamarix honey				Tamarix exorcism honey			
	25%	50%	75%	100%	25%	50%	75%	100%
Gram-Positive								
S. aureus	0 ± 0	12 ± 0.33*	15 ± 0.33*	19 ± 1.0	0 ± 0	0 ± 0	0 ± 00	17 ± 1.45
S. aureus (MRSA)	0 ± 0	0 ± 0	0 ± 0	16 ± 0.88	0 ± 0	11 ± 0.88*	14 ± 0.88*	18 ± 1.76
Gram-Negative								
E. coli	0 ± 0	0 ± 0	0 ± 0	11 ± 0.66	0 ± 0	0 ± 0	0 ± 0	12 ± 0.88
P.aeruginosa	0 ± 0	0 ± 0	0 ± 0	18 ± 0.58	16 ± 1.20*	18 ± 0.88*	19 ± 0.57*	21 ± 0.66
Fungi								
C. albicans	0 ± 0	0 ± 0	0 ± 0	13 ± 0.33*	0 ± 0	0 ± 0	0 ± 0	0 ± 0

*Significant of compared to two values in the same row.

Effect of N. sativa exorcism honey against the strains: The honey showed pronounced activity against P. aeruginosa, the maximum inhibition zone was shown at concentration of 100% as 21 ± 0.88 mm and 15 ± 0.66 mm at 50% concentration (Table 4) and figure1.

Table 4: Antimicrobial activities of the N. sativa honey before and after exorcism. **For each row,** Different letters after SE values indicate statistical significant differences between means (P < 0.05).

Organisms	Mean diameter of Inhibition Zone (mm) ± SE							
Honey	N. sativa honey				N. sativa exorcism honey			
	25%	50%	75%	100%	25%	50%	75%	100%
Gram-Positive								
S. aureus	0 ± 0	0 ± 0	0 ± 0	13 ± 0.88	0 ± 0	13 ± 0.88*	17 ± 0.88*	20 ± 0.88*
S. aureus (MRSA)	0 ± 0	0 ± 0	0 ± 0	0 ± 0	10 ± 0.57*	14 ± 2.40*	16 ± 2.40*	19 ± 2.08*
Gram-Negative								
E. coli	0 ± 0	0 ± 0	0 ± 0	12 ± 1.0	0 ± 0	0 ± 0	0 ± 0	18 ± 0.66*
P. aeruginosa	0 ± 0	0 ± 0	14 ± 0.58	18 ± 0.33	0 ± 0	15 ± 0.66*	18 ± 0.57	21 ± 0.88
Fungi								
C. albicans	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	10 ± 0.33*	12 ± 0.88*	15 ± 1.45*

*Significant of compared to two values in the same row.

N. sativa honey showed marked inhibition of growth on P. aeruginosa at concentration of 100% as 18 ± 0.33 mm and no activity against MRSA figure 1. S. aureus showed inhibition zone with N. sativa exorcism honey as 20 ± 0.88 and 13 ± 0.88 mm at 100% concentration with N. sativa honey figure2. Also showed that MRSA grow with inhibition zone at concentration of 100% as 19 ± 2.08 mm. C. albicans showed a little less inhibition zone with N. sativa exorcism honey which 15 ± 1.45 mm at 100%, 12 ± 0.88 mm at 75% and 10 ± 0.33 mm at 50% concentration and no effect was observed for all concentrations of N. sativa honey, compared to the N. sativa exorcism honey (Table 4).

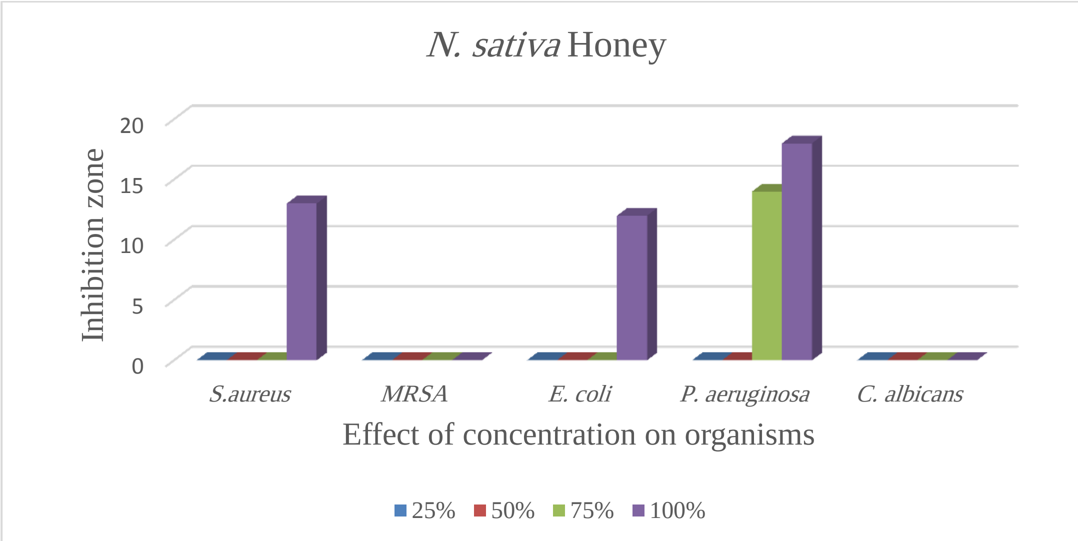

Figure1: The antibacterial activity of N. sativa honey compared with N. sativa.

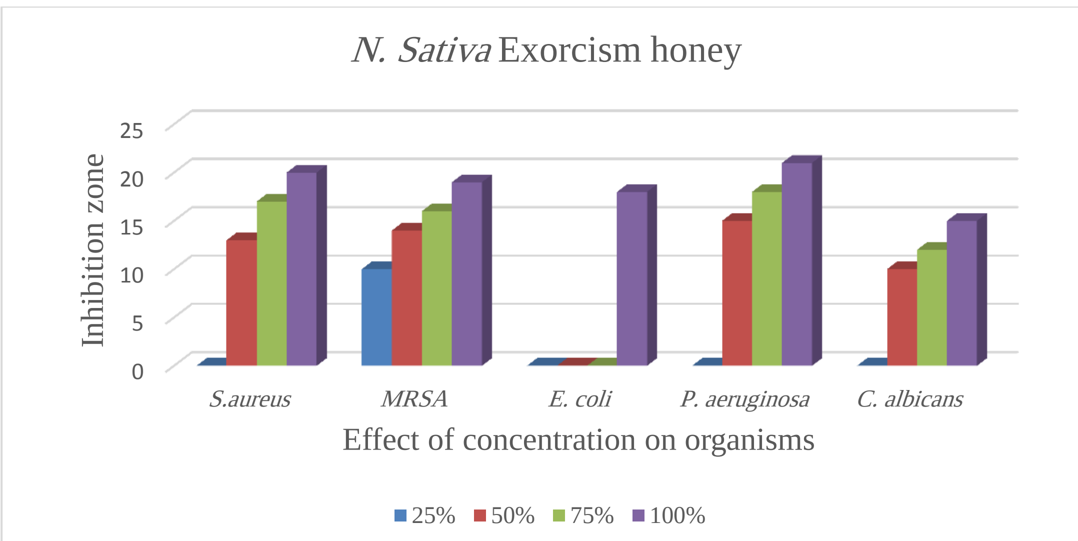

Figure2: The antibacterial activity of N. sativa exorcism honey .

Table 5 shows that only Spring exorcism honey has broad spectrum antifungal activity even in very low concentration 25% on C. albicans and the maximum inhibition zone was shown as 11 ± 0.58 mm.

In the case of E. coli there was no evidence of growth inhibition at concentration up to or including 75%, at concentration of 100% there was progressive increase in inhibition as honey concentration increased.

Table 5: Antimicrobial activities of the Spring honey before and after exorcism. **For each row,** Different letters after SE values indicate statistical significant differences between means ($P < 0.05$).

Organisms	Mean diameter of Inhibition Zone (mm) ± SE							
Honey	Spring honey				Spring exorcism honey			
	25%	50%	75%	100%	25%	50%	75%	100%
Gram-Positive								
S. aureus	0 ± 0	0 ± 0	15 ± 0.58	18 ± 0.58	0 ± 0	0 ± 0	17 ± 1.0	20 ± 0.57
S. aureus (MRSA)	0 ± 0	0 ± 0	0 ± 0	19 ± 0.88	0 ± 0	10 ± 0.57*	13 ± 0.33*	16 ± 0.66
Gram-Negative								
E. coli	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	15 ± 0.88
P. aeruginosa	0 ± 0	0 ± 0	10 ± 0.58*	14 ± 1.0	0 ± 0	0 ± 0	0 ± 0	12 ± 0.88
Fungi								
C. albicans	0 ± 0	10 ± 0.33	12 ± 0.33	15 ± 1.0	11 ± 0.58*	15 ± 0.58*	17 ± 0.88*	20 ± 0.88*

*Significant of compared to two values in the same row.

Zizphus honey before and after showed highly inhibition of growth and none of the organisms tested were resistance to the honey samples at concentration of 75% except E. coli (Table 6).

Upon comparison among the antimicrobial activities of the Zizphus honey samples before and after exorcism, highly statistical significant difference was found between their antimicrobial activity against MRSA and C. albicans. However, the Zizphus honey (before and after exorcism) have equipotent antibacterial against E. coli which was 14 ± 0.88 mm.

Table 6: Antimicrobial activities of the Zizphus honey before and after exorcism. **For each row,** Different letters after SE values indicate statistical significant differences between means ($P < 0.05$).

Organisms	Mean diameter of Inhibition Zone (mm) ± SE							
Honey	Zizphus honey				Zizphus exorcism honey			
	25%	50%	75%	100%	25%	50%	75%	100%
Gram-Positive								
S. aureus	0 ± 0	12 ± 0.58	15 ± 0.88	20 ± 1.15	11 ± 0.66*	14 ± 1.45	17 ± 1.0	20 ± 0.66
S. aureus (MRSA)	10 ± 0.33*	12 ± 0.58*	16 ± 1.76*	19 ± 1.86*	0 ± 0	0 ± 0	10 ± 0.88	14 ± 0.66
Gram-Negative								
E. coli	0 ± 0	0 ± 0	0 ± 0	14 ± 0.88	0 ± 0	0 ± 0	0 ± 0	14 ± 0.88
P. aeruginosa	0 ± 0	0 ± 0	11 ± 0.66*	15 ± 0.66	0 ± 0	0 ± 0	0 ± 0	13 ± 1.53
Fungi								
C. albicans	0 ± 0	0 ± 0	10 ± 0.33	14 ± 0.58	11 ± 0.33*	14 ± 0.33*	16 ± 0.66*	16 ± 0.58

*Significant of compared to two values in the same row.

In (Table 7) Myrrh exorcism honey was more susceptible against MRSA, the maximum inhibition zone was shown at concentration of 100% as 19 ± 0.88 mm. At concentration of 25%. In comparison to the all types of honey, only Myrrh and N. sativa exorcism honey were showed significant antibacterial activity on MRSA.

Table 7: Antimicrobial activities of the Myrrh honey before and after exorcism. **For each row,** Different letters after SE values indicate statistical significant differences between means (P < 0.05).

Organisms	Mean diameter of Inhibition Zone (mm) ± SE							
Honey	Myrrh honey				Myrrh exorcism honey			
	25%	50%	75%	100%	25%	50%	75%	100%
Gram-Positive								
S. aureus	0 ± 0	0 ± 0	10 ± 0.33*	14 ± 0.33	0 ± 0	0 ± 0	17 ± 0	15 ± 1.0
S. aureus (MRSA)	0 ± 0	0 ± 0	0 ± 0	15 ± 0.88	10 ± 0.33*	13 ± 0.33*	16 ± 0.66*	19 ± 0.88
Gram-Negative								
E. coli	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	13 ± 0.88*
P. aeruginosa	0 ± 0	0 ± 0	0 ± 0	12 ± 1.0	0 ± 0	0 ± 0	0 ± 0	11 ± 0.66
Fungi								
C. albicans	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	0 ± 0	15 ± 0.58*

*Significant of compared to two values in the same row.

The minimum inhibitory concentration of the six types of honey (before and after exorcism) against gram-positive bacteria are selected among the bacteria tested and results are shown in (Table 8). A lower MIC was observed for Tamarix exorcism honey 3.125% after 24 h incubation on MRSA, while the MBC was affected at concentration of 100% for the same honey. Tamarix honey was 50% against MRSA in comparison to Tamarix exorcism honey figure 2. In the case of S. aureus the lower MBC significantly was 6.25% by Zizphus exorcism honey. However, the MBC was 100% against MRSA. Generally, S. aureus was more susceptible to the six types of honey than the MRSA. Thymus honey exhibited no MBC values before and after exorcism toward S. aureus and MRSA (Table 8).

Finally, N. sativa honey (before and after exorcism) showed the same MIC and MBC values against S. aureus. While, significant difference were observed against MRSA at 100% concentration (P < 0.05).

Table 8: Minimum inhibitory concentration (MIC) and Minimum Bactericidal Concentration of all honey samples necessary to inhibit 100% of the microbial growth in vitro expressed in % V/V solution.

Strain / Honey samples	S. aureus		S. aureus (MRSA)	
	MIC	MBC	MIC	MBC
Thymus honey	-	-	100	-
Thymus exorcism honey	-	-	-	-
Tamarix honey	25	100	50	-
Tamarix exorcism honey	6.25	25	3.125	100
N. sativa honey	50	100	50	-
N. sativa exorcism honey	50	100	50	100
Spring honey	-	50	50	100
Spring exorcism honey	-	25	50	100
Zizphus honey	-	50	50	100
Zizphus exorcism honey	-	6.25	50	100
Myrrh honey	25	100	50	-
Myrrh exorcism honey	6.25	25	-	50

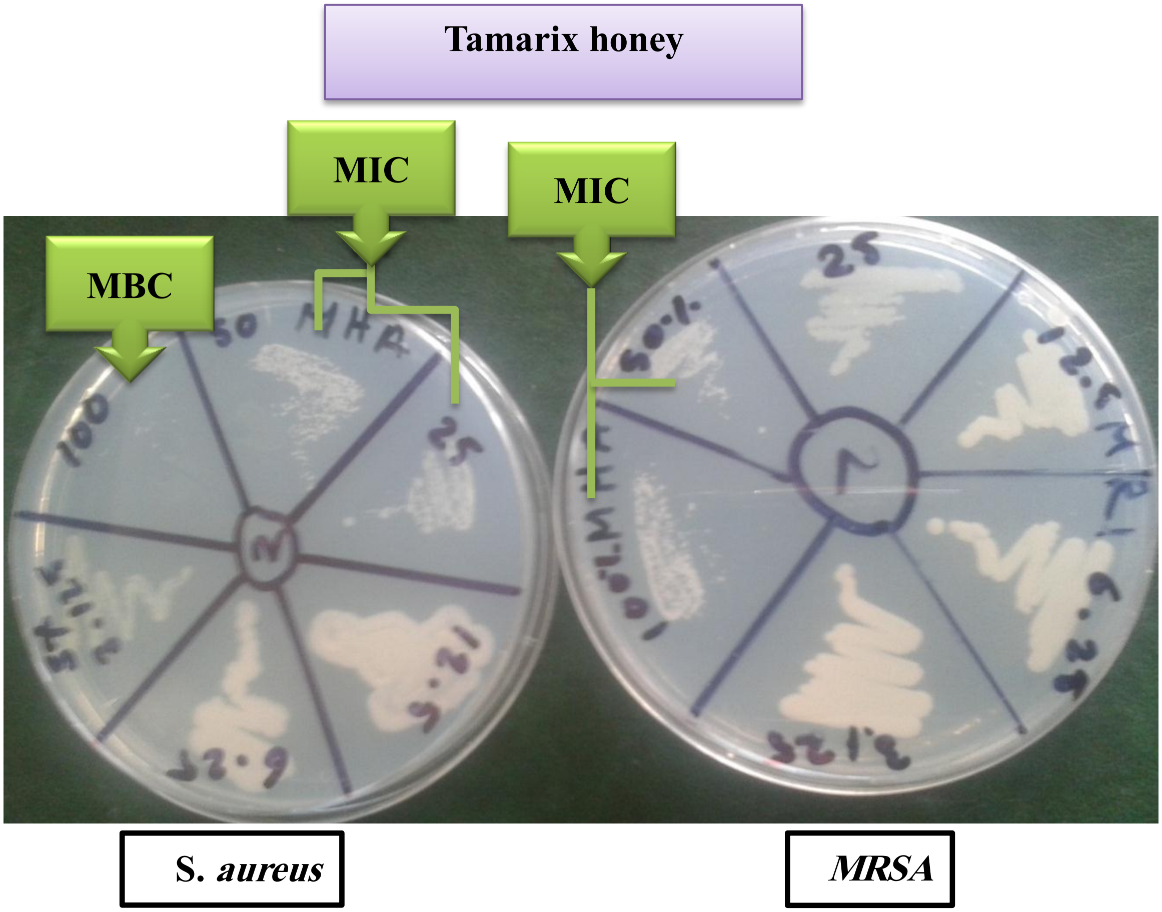

Figure2: The MIC and MBC activities of Tamarix honey against S. aureus and MRSA strains.

4. Discussion:

A total of six honey samples from different origins were evaluated for their antibacterial activity against selected bacteria and fungi species representing the Gram-positive species, S. aureus and MRSA, and the Gram negative species, P. aeruginosa and E. coli. And the fungi C. albicans. In general, as shown in results part. All tested honeys showed a measurable antibacterial activity against all of the tested bacteria with different values. Four of the tested bacteria were most sensitive to N. sativa exorcism honey comparable to other tested honeys showed a significant inhibition zone against Gram-positive bacteria, N. sativa, Thymus and Myrrh honey either showed inhibition to the tested fungi, especially. Spring honey displayed most a potent activity against C. albicans at concentration of 50 %. E. coli displayed the highest resistance for tested honeys. These data do not agree with the results reported by (Mohapatra et al. 2011) who showed that the Gram-negative bacteria are more susceptible to the inhibitory action of honey than are Gram-positive bacteria and agree with (Obeseiki-Ebor and Afonya 1984) and (Nzeako and Hamdi 2000) those found that ten honey samples investigation revealed that C. albicans sensitivity were less than other bacterial organisms tested and these are consistent with the data proved by our study. Also, In this study, honey sample showed the antimicrobial activity, and our result were in agreement with (Wi11ix et al. 1992) who found that honey inhibited the growth of Staphylococcus aureus, Escherichia coli, and Pseudomonas sp., and also in agreement with (Bilal et al., 1998) who found honey exhibited a fairly good antimicrobial activity against both Gram-negative and Gram-positive bacteria, and a remarkable activity was observed with Pseudomonas aeruginosa and Staphylococcus aureus.

The results shown by honey samples in relation to S. aureus may be important, given that in recent decades there has been a marked increase in difficult to treat skin and underlying tissue infections associated with S. aureus (Halcon and Milkus, 2004). It has been informed that S. aureus has developed resistance against several antibiotics and that it is the principal contaminant agent in many clinical infections (Moreno et al., 2005).

Thus, new strategies to treat wounds infected with S. aureus are needed, and the possibility to use honey appears as a convenient and less costly treatment option. Poor activity of the honeys against S. aureus was unexpected as previous reports by (Cooper et al. 1999). Part of the explanation for the difference in results from other studies may be due to methodological differences between studies because the agar dilution method used by these authors different from an agar well diffusion method that is used in this study.

However it is also likely to be due to variation in the natural floral origin of the honey being produced and variability in the performance of Mueller- Hinton agars from different manufacturers has been shown to be statistically significant, especially when testing E. coli (Barry AL and Effinger LJ 1974). The size of inoculums used, depth of medium in the plates, inoculation technique and, time period between inoculation and application of discs, incubation temperature and time of incubation will also cause differences in the results obtained (Acar JF and Goldstein FW 1996).

Our honey samples also exerted antimicrobial activities on P. aeruginosa, which were resistant to some antibiotics.

The MIC of honey which tested in this study had antimicrobial activity in the range between 3.125% - 100% against the tested microorganisms. The antibacterial potency differences among different studied honey samples could be attributed to the natural variations in floral sources of nectar and the different geographical locations since honey micro components possess physicochemical and phytochemical characteristics resulting in its potency that differs associated with botanical and geographical origins (Alzahrani, Alsabehi, Boukra, Abdellah, Bellik Y and et al 2012). Different honey samples of different botanical or geographical origins; Egyptian honey had MIC and MBC values as 12.5 and 50% v/v (Ali MWN; Abdel-Rahman M and Abdel-Hafeez MM 2005), Malysian honey as 5% and 6.25% w/v (Zainol M, Yusoff K and Yusof M 2013), UK Manuka honey had MIC as 6% w/v (Jenkis R and Cooper R 2012) and Ethiopian honey as 6.25% w/v (Ewnetu Y, Lemma W and Birhane N 2013). Honey antimicrobial action involves several mechanisms but mainly the presence of bacteriostatic and bactericidal action is due to production of hydrogen peroxide (FeÃ¡s X, Iglesias A, Rodrigues S and Estevinho L 2013). H_2O_2 alone may not be sufficient to the full activity (Chen C, Campbell LT, Blair SE and Carter DA 2012), since it is in conjunction with other unknown honey components produce bacterial cytotoxic effects and DNA degradation. Fortunately, S. aureus (either MRSA or methicillin sensitive) which is the most predominant and virulent pathogen was the most sensitive Staph. to honey antimicrobial action with highly significant. It is documented and proved that S. aureus was the most sensitive species to the antimicrobial activity of honey among all tested bacterial species studied (Andualem B 2013).

5. Conclusion:

The majority of the tested honeys exhibited inhibitory effects against different microorganisms. These results suggest that they might be used in treating a wide range of pathogenic Gram-positive, Gram-negative bacteria and fungi.

Abbreviations

ZOI:	Zone of inhibition
MIC:	Minimum Inhibitory Concentration
MBC:	Minimum Bactericidal Concentration
MHA:	Mueller Hinton Agar
MHB:	Mueller Hinton Broth
SDA:	Sabouraud Dextrose Agar
SDB:	Sabouraud Dextrose Broth
CIP:	Ciprofloxacin
KCA:	Ketoconazole
ATCC:	American Type Culture Collection

References:

Acar, J. F. & Goldstein, F. W. (1986). Disk susceptibility test. In Antibiotics in Laboratory Medicine, 2nd edn (Lorian, V., Ed.) pp. 50-2. Williams & Wilkins, Baltimore, MD**.**

Ali MWN; Abdel-Rahman M and Abdel-Hafeez MM (2005) Antibacterial activity of honey for treatment of subclinical bovine Mastitis: 1- In vitro study of bacterial inhibits and chemical bioassay of some different honeys. 8th Sci cong Egyptian society for cattle diseases Assiut Egypt 139-146.

Alzahrani H, Alsabehi R, BoukraÃ¢ L, Abdellah F, Bellik Y and et al (2012) Antibacterial and Antioxidant Potency of Floral Honeys from Different Botanical and Geographical Origins. Molecules 17: 10540-10549.

Andualem B (2013) Combined antibacterial activity of stingless bee (Apis mellipodae) honey and garlic (Allium sativum) extracts against standard and clinical pathogenic bacteria. Asian Pac J Trop Biomed 3: 725-731.

Barry AL and Effinger LJ. Performances of Mueller-Hinton agars prepared by three manufacturers, American J. Clin. Pathol, 1974, **62**, 113-117.

Barry, A. L. (1976): The antimicrobic susceptibility test: principles and practices. Lea and Febiger, Philadelphia.

Bergman, A., Yanai, J. , Weiss, J., Bell, D., and David, M. P. (1983). Acceleration of wound healing by topical application of honey. Am. J. Surg. 145: 374-376.

Bilal et al. Antimicrobial activity of honey on selected microorganisms: A preliminary study. Biomedical Research (India). 1998; 9: 51-54.

Chen C, Campbell LT, Blair SE and Carter DA (2012). The effect of standard heat and filtration processing procedures on antimicrobial activity and hydrogen peroxide levels in honey. Front Microbiol 3: 1â€“7.

Codex Alimentarius Commission (2001a) Codex Standard for Honey, FAO Rome. Alinorm 1: 19-26. ,

Codex Alimentarius Commission (2001b) Codex Standard 12, Revised Codex Standard for Honey, Standards and Standard Methods 11.

Cooper, R., Molan, P. and Harding, K. 1999. Antibacterial activity of honey against strains of Staphylococcus aureus from infected wounds. J. R. Soc. Med. 92, 283–285.

Elbagoury, E., and Rasomy, S. (1993). Antibacterial action of Manuka honey on anaerobic bacteriodes. Egypt. Dent. J. 39: 381-386.

Ewnetu Y, Lemma W and Birhane N (2013) Antibacterial effects of Apis mellifera and stingless bees honeys on susceptible and resistant strains of Escherichia coli, Staphylococcus aureus and Klebsiella pneumoniae in Gondar, Northwest Ethiopia. BMC Complementary and Alternative Medicine 13:269.

FeÃ¡s X, Iglesias A, Rodrigues S and Estevinho L (2013) Effect of Erica sp. honey against microorganisms of clinical importance: study of the factors underlying this biological activity. Molecules 18: 4233-4246

Halcon, L. and Milkus, K. 2004. Staphylococcus aureus and wounds: a review of tea tree oil as a promising antimicrobial. Am. J. Infect. Control 32, 402–408.

Jenkis R and Cooper R. 2012) Improving antibiotic activity against wound pathogens with manuka honey in vitro. PLoS One 7(9).

Medicine. 4th edition, edited by Victor L, Editor (Williams and Wilkins Publishers) 1996.

Mohapatra, D. – Thaku and V. – Brar 2011. Antibacterial efficacy of raw and processed honey. In Biotechnology research international, 2011, 6 p.

Molan PC. The antibacterial properties of honey. Chem in NZ, 1995; pp 10 – 14.

Molan, P., C. (1999). The role of honey in the management of wounds. J. Wound Care. 8:423-426.

Moreno, J., Cruz, C. and Renzon A. (2005). Tracking methicillin resistant Staphylococcus aureus clones Colombian hospitals over 7 years (1996–2003): emergence of a new dominant clone. Int. J. Antimicrobial Agents 25, 457–462.

Nzeako, B., Hamd and J., 2000. Antimicrobial potential of honey on some microbial isolates. SQU J. Sci. Res. Med. Sci. 2, 75–79.

Obaseiki-Ebor, E. E., and Afonya, T., C., A. (1984). In vitro evaluation of the anticandidiasis activity of honey distillate compared with that of some antimycotic agents. J. Pharmacy and Pharmacol. 36:283-284.

Wi11ix DJ, Molan PC and Harfoot CG. A comparision of the sensitivity of wound infecting species of bacteria to the antibacterial activity of Manuka honey and other honey. Journal of Applied Bacterial. 1992; 73 (5): 388-394.

Zainol M, Yusoff K and Yusof M (2013) Antibacterial activity of selected Malaysian honey. BMC Complementary and Alternative Medicine 13: 129.