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A New Application of Anuj Transform for Solving Partial Differential Equations

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الملخص:

في هذه الورقة، قمنا بتطبيق تحويل أونج التكاملي على بعض المعادلات التفاضلية الجزئية والحصول على حلها وذلك بعد استنتاج المشتقات الجزئية لتحويل أونج.

كلمات مفتاحية: تحويل أونج، المعادلات التفاضلية الجزئية.

Abstract:

In this paper, we have applied Anuj transform to solve partial differential equations and derived the partial derivatives of the Anuj transform.

Keywords: Anuj Transform, Partial Differential Equations.

Introduction:

Partial differential equations (PDE) are one of the most important areas of mathematics science in general, applied mathematics in particular. This led to invention of many methods and techniques for solving PDE [2], [3], [4] & [5]. The integral transforms were and still are one of the most popular methods for solving PDE such as Sumudu transform, Kamal transform, Laplace transform and Mohand transform to name but a few. Recently, a new transform has appeared which named Anuj Transform [1]. In this paper, we will apply the Anuj transform to solve PDEs. Therefore, our target is to derive the partial derivatives of the Anuj transform and solve some problems as examples.

1. DEFINITION OF ANUJ TRANSFORM [1]

If we have a real function $f(t); t \geq 0$, then Anuj Transform is defined as the following integral equation:

$$\Lambda\{f(t)\} = \nu^2 \int_0^\infty f(t)e^{-\frac{t}{\nu}}dt = \Re(\nu)\,,\ \nu > 0 \qquad (1)$$

where Λ is the Anuj Transform operator and ν is real parameter.

2. Anuj Transform and Inverse Anuj Transform of Some Functions [1]

S. No.	$f(t)$	$\Lambda\{f(t)\} = \Re(p)$	$\Re(p)$	$f(t) = \Lambda^{-1}\{\Re(p)\}$
1	1	v^3	v^3	1
2	t	v^4	v^4	t
3	t^2	$2!\,v^5$	v^5	$\frac{t^2}{2!}$
4	$t^m\,; m \in \mathbb{N}$	$m!\,v^{m+3}$	$v^m\,;\ m \in \mathbb{N}$	$\frac{t^{m-3}}{(m-3)!}$
5	$t^m\,; m > -1$	$\Gamma(m+1)v^{m+3}$	$v^{m+3}; m > -1$	$\frac{t^m}{\Gamma(m+1)}$
6	e^{at}	$\frac{v^3}{1-av}$	$\frac{v^3}{1-av}$	e^{at}
7	$\sin\beta t$	$\frac{\beta v^4}{1+\beta^2 v^2}$	$\frac{\beta v^4}{1+\beta^2 v^2}$	$\frac{\sin\beta t}{\beta}$
8	$\cos\beta t$	$\frac{v^3}{1+\beta^2 v^2}$	$\frac{v^3}{1+\beta^2 v^2}$	$\cos\beta t$
9	$\sinh\beta t$	$\frac{\beta v^4}{1-\beta^2 v^2}$	$\frac{v^4}{1-\beta^2 v^2}$	$\frac{\sinh\beta t}{\beta}$
10	$\cosh\beta t$	$\frac{v^3}{1-\beta^2 v^2}$	$\frac{v^3}{1-\beta^2 v^2}$	$\cosh\beta t$
11	$e^{at}f(t)$	$(1-ap)^2\Re\left(\frac{v}{1-av}\right)$	-	-

Table (1): Anuj transform and inverse Anuj transform of some functions.

3. Anuj Transform of Partial Derivatives:

To obtain Anuj transform of partial derivatives we have:

$$\Lambda\{f(t)\} = v^2 \int_0^\infty f(t)e^{-\frac{t}{v}}dt = \Re(v)$$

and let us to consider $f(t) = \frac{\partial f(x,t)}{\partial t}$, then: $\Lambda\left\{\frac{\partial f(x,t)}{\partial t}\right\} = v^2 \int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-\frac{t}{v}}dt$. Now, we will use integration by parts as follows:

$$\Lambda\left\{\frac{\partial f(x,t)}{\partial t}\right\} = v^2 \int_0^\infty \frac{\partial f(x,t)}{\partial t} e^{-\frac{t}{v}}dt = \lim_{q\to\infty} v^2 \int_0^q \frac{\partial f(x,t)}{\partial t} e^{-\frac{t}{v}}dt$$

$$= v^2 \lim_{q\to\infty}\left[\left[f(x,t)e^{-\frac{t}{v}}\right]_0^q + \frac{1}{v}\int_0^q f(x,t)e^{-\frac{t}{v}}dt\right]$$

$$= -\nu^2 f(x,0) + \frac{1}{\nu}\Re(x,\nu)$$

$$\therefore \Lambda\left\{\frac{\partial f(x,t)}{\partial t}\right\} = \nu^{-1}\Re(x,\nu) - \nu^2 f(x,0) \qquad (2)$$

To find $\Lambda\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = \nu^2 \int_0^\infty \frac{\partial^2 f(x,t)}{\partial t^2} e^{-\frac{t}{\nu}} dt$ and in the same way above, we will get:

$$\Lambda\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = \nu^2 \int_0^\infty \frac{\partial^2 f(x,t)}{\partial t^2} e^{-\frac{t}{\nu}} dt = \lim_{q\to\infty} \nu^2 \int_0^q \frac{\partial^2 f(x,t)}{\partial t^2} e^{-\frac{t}{\nu}} dt$$

$$= \nu^2 \lim_{q\to\infty}\left[\left[\frac{\partial f(x,t)}{\partial t} e^{-\frac{t}{\nu}}\right]_0^q + \frac{1}{\nu}\int_0^q \frac{\partial f(x,t)}{\partial t} e^{-\frac{t}{\nu}} dt\right]$$

By using equation (2), we obtain:

$$\Lambda\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = \nu^{-2}\Re(x,\nu) - \nu f(x,0) - \nu^2 \frac{\partial f(x,0)}{\partial t} \qquad (3)$$

Thus, we can deduce the *nth* partial derivative in the following formula:

$$\Lambda\left\{\frac{\partial^n f(x,t)}{\partial t^n}\right\} = \nu^{-n}\Re(x,\nu) - \nu^{-n+3} f(x,0) - \nu^{-n+4}\frac{\partial f(x,0)}{\partial t} - \nu^{-n+5}\frac{\partial^2 f(x,0)}{\partial t^2} - \nu^{-n+6}\frac{\partial^3 f(x,0)}{\partial t^3} - \cdots - \nu\frac{\partial^{n-2} f(x,0)}{\partial t^{n-2}} - \nu^2 \frac{\partial^{n-1} f(x,0)}{\partial t^{n-1}} \qquad (4)$$

Now, if we have $\Lambda\left\{\frac{\partial f(x,t)}{\partial x}\right\} = \nu^2 \int_0^\infty \frac{\partial f(x,t)}{\partial x} e^{-\frac{t}{\nu}} dt$, and by using the Leibinz rule and respect to the parameter x, we should get:

$$\Lambda\left\{\frac{\partial f(x,t)}{\partial x}\right\} = \nu^2 \int_0^\infty \frac{\partial f(x,t)}{\partial x} e^{-\frac{t}{\nu}} dt = \frac{\partial}{\partial x}\nu^2 \int_0^\infty f(x,t) e^{-\frac{t}{\nu}} dt$$

$$\therefore \Lambda\left\{\frac{\partial f(x,t)}{\partial x}\right\} = \frac{d}{dx}[\Re(x,\nu)] \qquad (5)$$

Also we can deduce:

$$\Lambda\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} = \frac{d^2}{dx^2}[\Re(x,\nu)] \qquad (6)$$

In the general formula, we have:

$$\Lambda\left\{\frac{\partial^n f(x,t)}{\partial x^n}\right\} = \frac{d^n}{dx^n}[\Re(x,\nu)] \quad (7)$$

Therefore, we can summarize the above work in table (2):

S. No.	$f(t)$	$\Lambda\{f(t)\}$
1	$\frac{\partial f(x,t)}{\partial t}$	$\nu^{-1}\Re(x,\nu) - \nu^2 f(x,0)$
2	$\frac{\partial^2 f(x,t)}{\partial t^2}$	$\nu^{-2}\Re(x,\nu) - \nu f(x,0) - \nu^2 \frac{\partial f(x,0)}{\partial t}$
3	$\frac{\partial^n f(x,t)}{\partial t^n}$	$\nu^{-n}\Re(x,\nu) - \nu^{-n+3} f(x,0) - \nu^{-n+4}\frac{\partial f(x,0)}{\partial t} - \nu^{-n+5}\frac{\partial^2 f(x,0)}{\partial t^2} - \nu^{-n+6}\frac{\partial^3 f(x,0)}{\partial t^3} - \cdots - \nu\frac{\partial^{n-2} f(x,0)}{\partial t^{n-2}} - \nu^2\frac{\partial^{n-1} f(x,0)}{\partial t^{n-1}}$
4	$\frac{\partial f(x,t)}{\partial x}$	$\frac{d}{dx}[\Re(x,\nu)]$
5	$\frac{\partial^2 f(x,t)}{\partial x^2}$	$\frac{d^2}{dx^2}[\Re(x,\nu)]$
6	$\frac{\partial^n f(x,t)}{\partial x^n}$	$\frac{d^n}{dx^n}[\Re(x,\nu)]$

Table (2)**:** Anuj Transform of partial derivatives.

4. Applications

Example (1): Solve the partial differential equation:

$$2u_t + 3u_{tt} = 3u_{xx} + 2(x^2 - 9x - t + 3) \quad \text{(I)}$$

with boundary conditions:

$u(x,0) = x^3 + x, u_t(x,0) = x^2,\ u(0,t) = t^2$ & $u_x(0,t) = 1.$

Solution: Taking Anuj transform to both sides of given equation (I) and using table (2), we will get:

$$2\Lambda\{u_t\} + 3\Lambda\{u_{tt}\} = 3\Lambda\{w_{xx}\} + 2(x^2 - 9x + 3)\Lambda\{1\} - 2\Lambda\{t\}$$

$$2[\nu^{-1}\Re(x,\nu) - \nu^2 u(x,0)] + 3[\nu^{-2}\Re(x,\nu) - \nu u(x,0) - \nu^2 u_t(x,0)] = 3\Re_{xx}(x,\nu) + 2\nu^3(x^2 - 9x + 3) - 2\nu^4$$

$$\Rightarrow 3\Re_{xx}(x,\nu) - \frac{2\nu+3}{\nu^2}\Re(x,\nu) = 2\nu^4 - 2\nu^3(x^2 - 9x + 3) - 3\nu^2 x^2 - (2\nu^2 + 3\nu)(x^3 + x) \quad \text{(II)}$$

Last equation is a second-order differential equation (ODE) in x, where it has complementary solution ($\Re_c$):

$$\Re_c(x,\nu) = C_1 e^{\alpha x} + C_2 e^{-\alpha x} \,;\, \alpha = \frac{\sqrt{2\nu+3}}{\nu\sqrt{3}}$$

To obtain the particular solution ($\Re_p$) we consider:

$$\Re_p(x,\nu) = Ax^3 + Bx^2 + Cx + D \Rightarrow \frac{d^2\,\Re_p(x,\nu)}{dx^2} = 6Ax + 2B$$

and substitute into Eq. (II) we get:

$$3[6Ax + 2B] - \frac{2\nu+3}{\nu^2}[Ax^3 + Bx^2 + Cx + D] = 2\nu^4 - 2\nu^3(x^2 - 9x + 3) - 3\nu^2x^2 - (2\nu^2 + 3\nu)(x^3 + x)$$

$$\left(6B - \frac{2\nu+3}{\nu^2}D\right) + \left(18A - \frac{2\nu+3}{\nu^2}C\right)x - \frac{2\nu+3}{\nu^2}Bx^2 - \frac{2\nu+3}{\nu^2}Ax^3 = 2\nu^4 - 6\nu^3 - x(2\nu^2 + 3\nu - 18\nu^3) - x^3(2\nu^3 + 3\nu^2) - x^2(2\nu^3 + 3\nu^2)$$

which gives us the following values:

$A = \nu^3$, $B = \nu^4$, $C = \nu^3$ & $D = 2\nu^5$.

$$\Rightarrow \quad \Re(x,\nu) = C_1 e^{\alpha x} + C_2 e^{-\alpha x} + \nu^3x^3 + \nu^4x^2 + \nu^3x + 2\nu^5; \alpha = \frac{\sqrt{2\nu+3}}{\nu\sqrt{3}}.$$

As we have: $u(0,t) = t^2 \Rightarrow \Re(0,\nu) = 2\nu^5$

$$\Rightarrow \quad \Re(0,\nu) = C_1 + C_2 + 2\nu^5 = 2\nu^5 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_1 = -C_2.$$

And: $u_x(0,t) = 1 \Rightarrow \Re_x(0,\nu) = \nu^3$

$$\Rightarrow \quad \Re_x(0,\nu) = \frac{\sqrt{2\nu+3}}{\nu\sqrt{3}}C_1 - \frac{\sqrt{2\nu+3}}{\nu\sqrt{3}}C_2 + \nu^3 = \nu^3 \Rightarrow C_1 - C_2 = 0 \Rightarrow C_1 = C_2 = 0.$$

$$\therefore \quad \Re(x,\nu) = \nu^3x^3 + \nu^4x^2 + \nu^3x + 2\nu^5 \qquad \text{(III)}$$

Now, we take the inverse Anuj transform to Eq. (III),

$$\Lambda\{\Re(x,v)\} = x^3\Lambda\{v^3\} + x^2\Lambda\{v^4\} + x\Lambda\{v^3\} + \Lambda\{2v^5\}$$

then the solution of Eq. (I) is:

$$u(x,t) = x^3 + tx^2 + x + t^2.$$

Example (2): Solve the partial differential equation:

$$w_{tt} - w_{xx} = 0\,;\ \ 0 \le x \le \pi\ \ , t \ge 0, \tag{I}$$

with boundary conditions $w(x,0) = \sin x$, $w_t(x,0) = 0, w(0,t) = 0$ & $w(\pi,t) = 0$.

Solution: By applying Anuj transform to both sides of Eq. (I) and using table (2), we get:

$$\Lambda\{w_{tt}\} - \Lambda\{w_{xx}\} = 0$$

$$v^{-2}\Re(x,v) - vw(x,0) - v^2 w_t(x,0) - \Re_{xx}(x,v) = 0$$

$$\Rightarrow \Re_{xx}(x,v) - v^{-2}\Re(x,v) = -vw(x,0) - v^2 w_t(x,0)$$

The boundary conditions $w(0,t) = w(\pi,t) = 0$ give a zero solution to the homogeneous differential equation of Eq. (I). For inhomogeneous differential equation and by using boundary conditions $w(x,0) = \sin x$ & $w_t(x,0) = 0$, we obtain:

$$\Re_{xx}(x,v) - v^{-2}\Re(x,v) = -v\sin x$$

$$\Re(x,v) = \frac{-v}{D^2 - \frac{1}{v^2}}(\sin x)\ = \sin x\,\frac{-v}{-1-\frac{1}{v^2}}$$

$$\Re(x,v) = \sin x\,\frac{v^3}{1+v^2} \tag{II}$$

Now, by applying the inverse Anuj transform to Eq. (II) we obtain the solution:

$$w(x,t) = \sin x\cos t.$$

Example (3): Solve the partial differential equation:

$$w_x(x,t) - 2w_t(x,t) = w(x,t)\,;\ \ x > 0\ \ , t > 0, \tag{I}$$

with initial conditions $w(x,0)=e^{-3x}, w(0,t)=e^{-2t}$.

Solution: By applying Anuj transform to both sides of Eq. (I), we get:

$$\Re_x(x,\nu)-2[\nu^{-1}\Re(x,\nu)-\nu^2 w(x,0)]=\Re(x,\nu)$$

$$\Rightarrow \Re_x(x,\nu)-\left(\frac{2}{\nu}+1\right)\Re(x,\nu)=-2\nu^2 e^{-3x}. \quad \text{(II)}$$

Eq. (II) is the linear ODE and it has the integration factor: $\lambda=e^{-\left(\frac{2}{\nu}+1\right)x}$. Thus:

$$\Re(x,\nu)=\frac{\nu^3}{1+2\nu}e^{-3x}+Ce^{\left(\frac{2}{\nu}+1\right)x} \quad \text{(III)}$$

Now, we have: $w(0,t)=e^{-2t}\Rightarrow \Re(0,\nu)=\frac{\nu^3}{1+2\nu}$

substitute into Eq. (III) we get:

$$\frac{\nu^3}{1+2\nu}=\frac{\nu^3}{1+2\nu}+C \;\Rightarrow\; C=0$$

$$\therefore \quad \Re(x,\nu)=\frac{\nu^3}{1+2\nu}e^{-3x}. \quad \text{(VI)}$$

By applying the inverse Anuj transform to Eq. (VI) we obtain the solution:

$$w(x,t)=e^{-(2t+3x)}.$$

Example (4): Solve the partial differential equation:

$$w_{xx}=w_t \quad \text{(I)}$$

with boundary conditions $w(x,0)=\sin\frac{\pi}{\lambda}x, w(0,t)=0$ & $w(\lambda,t)=0$.

Solution: By applying Anuj transform to both sides of Eq. (I), we get:

$$\Lambda\{w_{xx}\}=\Lambda\{w_t\}$$

$$\Re_{xx}(x,\nu)=\nu^{-1}\Re(x,\nu)-\nu^2 w(x,0)$$

$$\therefore \Re_{xx}(x,\nu)-\frac{1}{\nu}\Re(x,\nu)=-\nu^{-2}\sin\frac{\pi}{\lambda}x. \qquad \text{(II)}$$

Eq. (II) is a second ODE in x, where it has complementary solution:

$$\Re_c(x,\nu)=C_1e^{\frac{1}{\sqrt{\nu}}x}+C_2e^{-\frac{1}{\sqrt{\nu}}x}.$$

To obtain the particular solution we have:

$$\Re_p(x,\nu)=\frac{-\nu^2}{D^2-\frac{1}{\nu}}\left(\sin\frac{\pi}{\lambda}x\right)$$

$$\Rightarrow \Re_p(x,\nu)=\sin\frac{\pi}{\lambda}x\frac{\nu^3}{1+\frac{\pi^2}{\lambda^2}\nu}$$

$$\therefore \quad \Re(x,\nu)=C_1e^{\frac{1}{\sqrt{\nu}}x}+C_2e^{-\frac{1}{\sqrt{\nu}}x}+\sin\frac{\pi}{\lambda}x\frac{\nu^3}{1+\frac{\pi^2}{\lambda^2}\nu}.$$

As we have: $w(0,t)=w(\lambda,t)=0$

$$\therefore \quad w(0,t)=0\Rightarrow C_1+C_2=0\Rightarrow C_2=-C_1.$$

$$\& \quad w(\lambda,t)=0\Rightarrow C_1e^{\frac{1}{\sqrt{\nu}}x}+C_2e^{-\frac{1}{\sqrt{\nu}}x}=0\Rightarrow C_1\left(e^{\frac{1}{\sqrt{\nu}}x}-e^{-\frac{1}{\sqrt{\nu}}x}\right)=0.$$

As: $e^{\frac{1}{\sqrt{\nu}}x}-e^{-\frac{1}{\sqrt{\nu}}x}\neq 0\Rightarrow C_1=0\Rightarrow C_2=0.$

$$\therefore \quad \Re(x,\nu)=\sin\frac{\pi}{\lambda}x\frac{\nu^3}{1+\frac{\pi^2}{\lambda^2}\nu} \qquad \text{(III)}$$

Now, we take the inverse Anuj transform to Eq. (III)

$$\Lambda\{\Re(x,\nu)\}=\sin\frac{\pi}{\lambda}x\,\Lambda\left\{\frac{\nu^3}{1+\frac{\pi^2}{\lambda^2}\nu}\right\}$$

then the solution of Eq. (I) is:

$$u(x,t)=e^{-\frac{\pi^2}{\lambda^2}t}\sin\frac{\pi}{\lambda}x.$$

Conclusion: In this paper, authors have successfully appled the application of Anuj transform to solve PDE's and the examples given in the paper have been solved. The results of problems show that the Anuj transform is very useful integral transform for solving such equations. In addition, all the obtained solutions of the indicated problems are satisfied by putting them back in the corresponding equations. Anuj transform can be used in future to solve a wide class of similar equations.

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